True or False

1. T, by Fact 7.2.2

2. T, by Definition 7.2.3

3. F; If $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then eigenvalue 1 has geometric multiplicity 1 and algebraic multiplicity. 2.

4. T, by Fact 7.4.3

5. T; $A = AI_n = A[\vec{e}_1 \ldots \vec{e}_n] = [\lambda_1\vec{e}_1 \ldots \lambda_n\vec{e}_n]$ is diagonal.

6. T; If $A\vec{v} = \lambda\vec{v}$, then $A^3\vec{v} = \lambda^3\vec{v}$.

7. T; Consider a diagonal 5×5 matrix with only two distinct diagonal entries.

8. F, by Fact 7.2.7.

9. T, by Summary 7.1.4

10. T, by Fact 7.2.4

11. F; Consider $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

12. F; Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, $\alpha = 2, B = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}$, $\beta = 5$, for example. Then $\alpha\beta = 10$ isn't an eigenvalue of $AB = \begin{bmatrix} 8 & 0 \\ 0 & 15 \end{bmatrix}$.

13. T; If $A\vec{v} = 3\vec{v}$, then $A^2\vec{v} = 9\vec{v}$.

14. T; Construct an eigenbasis by combining a basis of V with a basis of $V^\perp$.

15. T, by Fact 7.5.5

16. F; Let $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, for example.

17. T, by Example 6 of Section 7.5

18. T; The geometric multiplicity of eigenvalue 0 is $\dim(\ker A) = n - \mathrm{rank}(A)$.

19. T; If $S^{-1}AS = D$, then $S^T A^T (S^T)^{-1} = D$.

20. F; Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, for example.

21. F; Consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, with $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

22. T, by Fact 7.5.5

23. F; Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, for example.

24. F; Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, with $AB = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$, for example.

25. T; If $S^{-1}AS = D$, then $S^{-1}A^{-1}S = D^{-1}$ is diagonal

26. F; the equation $\det(A) = \det(A^T)$ holds for all square matrices, by Fact 6.2.1

27. T; The sole eigenvalue, 7, must have geometric multiplicity 3.

28. F; Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, with $A + B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, for example.

29. F; Consider the zero matrix.

30. T; If $A\vec{v} = \alpha\vec{v}$ and $B\vec{v} = \beta\vec{v}$, then $(A + B)\vec{v} = A\vec{v} + B\vec{v} = \alpha\vec{v} + \beta\vec{v} = (\alpha + \beta)\vec{v}$.

31. F; Consider the identity matrix.

32. T; Both A and B are similar to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, by Fact 7.4.1

33. F; Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, for example.

34. F; Consider $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

35. F; Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, for example.

36. T; A nonzero vector on L and a nonzero vector on $L^\perp$ form an eigenbasis.

37. T; The eigenvalues are 3 and -2.

38. T, by Fact 7.3.7

39. T, by Fact 7.3.5

40. T; An eigenbasis for A is an eigenbasis for $A + 4I_4$ as well.

41. F; Consider a rotation through $\pi/2$.

42. T; Suppose $\begin{bmatrix} A & A \\ 0 & A \end{bmatrix} \begin{bmatrix} \vec{v} \\ \vec{w} \end{bmatrix} = \begin{bmatrix} A(\vec{v} + \vec{w}) \\ A\vec{w} \end{bmatrix} = \begin{bmatrix} \lambda \vec{v} \\ \lambda \vec{w} \end{bmatrix}$ for a nonzero vector $\begin{bmatrix} \vec{v} \\ \vec{w} \end{bmatrix}$. If $\vec{w}$ is nonzero, then it is an eigenvector of A with eigenvalue λ; otherwise $\vec{v}$ is such an eigenvector.

43. F; Consider $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

44. T; Note that $S^{-1}AS = D$, so that $D^4 = S^{-1}A^4S = S^{-1}0S = 0$, and therefore $D = 0$ (since D is diagonal) and $A = SDS^{-1} = 0$.

45. T; There is an eigenbasis $\vec{v}_1, \ldots, \vec{v}_n$, and we can write $\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$. The vectors $c_i\vec{v}_i$ are either eigenvectors or zero.

46. T; If $A\vec{v} = \alpha\vec{v}$ and $B\vec{v} = \beta\vec{v}$, then $AB\vec{v} = \alpha\beta\vec{v}$.

47. T, by Fact 7.3.8

48. F; Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, for example.

49. T; Recall that the rank is the dimension of the image. If $\vec{v}$ is in the image of A, then $A\vec{v}$ is in the image of A as well, so that $A\vec{v}$ is parallel to $\vec{v}$.

50. F; Consider $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

51. T; If $A\vec{v} = \lambda\vec{v}$ for a nonzero $\vec{v}$, then $A^4\vec{v} = \lambda^4\vec{v} = \vec{0}$, so that $\lambda^4 = 0$ and $\lambda = 0$.

52. F; Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, for example.

53. T; If the eigenvalue associated with $\vec{v}$ is $\lambda = 0$, then $A\vec{v} = \vec{0}$, so that $\vec{v}$ is in the kernel of A; otherwise $\vec{v} = A\left(\frac{1}{\lambda}\vec{v}\right)$, so that $\vec{v}$ is in the image of A.

54. T; either there are two distinct real eigenvalues, or the matrix is of the form kI_2.

55. T; Either $A\vec{u} = 3\vec{u}$ or $A\vec{u} = 4\vec{u}$.

56. T; Note that $(\vec{u}\vec{u}^T)\vec{u} = \|\vec{u}\|^2\vec{u}$.

57. T; Suppose $A\vec{v}_i = \alpha_i \vec{v}_i$ and $B\vec{v}_i = \beta_i \vec{v}_i$, and let $S = [\vec{v}_1 \ldots \vec{v}_n]$. Then $ABS = BAS = [\alpha_1 \beta_1 \vec{v}_1 \ldots \alpha_n \beta_n \vec{v}_n]$, so that $AB = BA$.

58. T; Note that a nonzero vector $\vec{v} = \begin{bmatrix} p \\ q \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ if (and only if) $A\vec{v} = \begin{bmatrix} ap + bq \\ cp + dq \end{bmatrix}$ is parallel to $\vec{v} = \begin{bmatrix} p \\ q \end{bmatrix}$, that is, if $\det \begin{bmatrix} p & ap + bq \\ q & cp + dq \end{bmatrix} = 0$. Check that this is the case if (and only if) $\vec{v}$ is an eigenvector of $\mathrm{adj}(A)$ (use the same criterion).