True or False

1. T, by the spectral theorem (Fact 8.1.1)

2. T. Note that $\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = a > 0$, by Definition 8.2.3.

3. F. The orthogonal matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ fails to be diagonalizable (over $\mathbb{R}$).

4. T. If $A = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, then the eigenvalue of $A^T A = \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = [25]$ is $\lambda = 25$, so that the singular value of A is $\sigma = \sqrt{\lambda} = 5$.

5. F. The last term, $5x_2$, does not have the form required in Definition 8.2.1

6. F. The singular values of A are *the square roots of* the eigenvalues of $A^T A$, by Definition 8.3.1.

7. T, by Fact 8.2.4.

8. T, by Definition 8.2.1

9. T. If $D = \begin{bmatrix} \lambda_1 & \cdot & 0 \\ \cdot & \cdot & \cdot \\ 0 & \cdot & \lambda_n \end{bmatrix}$, then $D^T D = D^2 = \begin{bmatrix} \lambda_1^2 & \cdot & 0 \\ \cdot & \cdot & \cdot \\ 0 & \cdot & \lambda_n^2 \end{bmatrix}$. The eigenvalues of $D^T D$ are $\lambda_1^2, \dots, \lambda_n^2$, and the singular values of D are $\sqrt{\lambda_1^2} = |\lambda_1|, \dots, \sqrt{\lambda_n^2} = |\lambda_n|$.

10. F, since $\det \begin{bmatrix} 2 & \frac{5}{2} \\ \frac{5}{2} & 3 \end{bmatrix} = -\frac{1}{4} < 0$ (see Fact 8.2.7).

11. F. Consider $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

12. T, since AA^T is symmetric (use the spectral theorem)

13. T, by Fact 8.2.4: all the eigenvalues are positive.

14. T, since the matrix is symmetric.

15. F. Consider the shear matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. The unit circle isn't mapped into itself, so that the singular values fail to be 1, 1.

16. F. In general, $(A^T A)^T = A^T A \neq AA^T$

17. T, by Fact 8.3.2

18. T. All four eigenvalues are negative, so that their product, the determinant, is positive.

19. T, by Fact 8.1.2

20. F, since the determinant is 0, so that 0 is an eigenvalue.

21. F. As a counterexample, consider $A = B = 2I_n$.

22. T, since $\vec{e}_i^T A \vec{e}_i = a_{ii} < 0$.

23. T. By Fact 7.3.8, matrices A and B have the same eigenvalues. Now use Fact 8.2.4.

24. T. The spectral theorem guarantees that there is an orthogonal R such that $R^T AR$ is diagonal. Now let $S = R^T$.

25. F. Let $A = I_2$, $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

26. T. Consider the singular value decomposition $A = U \sum V^T$, or $AV = U \sum$, where V is orthogonal (see Fact 8.3.5). We can let $S = V$, since the columns of $AS = AV = U \sum$ are orthogonal, by construction.

27. T. By the spectral theorem, A is diagonalizable: $S^{-1}AS = D$ for some invertible S and a diagonal D. Now $D^n = S^{-1}A^n S = S^{-1}0S = 0$, so that $D = 0$ (since D is diagonal). Finally, $A = SDS^{-1} = S0S^{-1} = 0$, as claimed.

28. F. If k is negative, then $kq(\vec{x})$ will be negative definite.

29. T. The eigenvalues $\lambda_1, \ldots, \lambda_n$ of A are nonzero, since A is invertible, so that the eigenvalues $\lambda_1^2, \ldots, \lambda_n^2$ of A^2 are positive. Now use Fact 8.2.4.

30. T, by Fact 8.3.2, since $\vec{v} = A\vec{e}_1$ and $\vec{w} = A\vec{e}_2$ are the principal semi-axes of the image of the unit circle.

31. F. For example, $(x_1^2)(x_2x_3)$ fails to be a quadratic form.

32. T. We can write $q(\vec{x}) = \vec{x}^T \begin{bmatrix} 1 & \frac{5}{2} \\ \frac{5}{2} & 4 \end{bmatrix} \vec{x}$.

33. F. Consider $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, which is indefinite.

34. T, by Definition 8.2.3: $\vec{x}^T(A+B)\vec{x} = \vec{x}^T A\vec{x} + \vec{x}^T B\vec{x} > 0$ for all nonzero $\vec{x}$.

35. T, since $\vec{x} \cdot A\vec{x}$ is positive, so that $\cos\alpha$ is positive, where α is the angle enclosed by $\vec{x}$ and $A\vec{x}$.

36. T. Preliminary remark: If σ is the largest singular value of an $m \times n$ matrix M, then $\|M\vec{v}\| \le \sigma\|\vec{v}\|$ for all $\vec{v}$ in $\mathbb{R}^n$ (see Exercise 8.3.26). Now let σ_1, σ_2 be the singular values of matrix AB, with $\sigma_1 \ge \sigma_2$, and let $\vec{v}_1$ be a unit vector in $\mathbb{R}^2$ such that $\|AB\vec{v}_1\| = \sigma_1$ (see Fact 8.3.3). Now $\sigma_2 \le \sigma_1 = \|A(B\vec{v}_1)\| \le 3\|B\vec{v}_1\| \le 3 \cdot 5\|\vec{v}_1\| = 15$, proving our claim; note that we have used the preliminary remark twice.

37. F. Consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

38. T. If λ is the smallest eigenvalue of A, let $k = 1 - \lambda$. Then the smallest eigenvalue of $A + kI_n$ is $\lambda + k = 1$, so that all the eigenvalues of $A + kI_n$ are positive. Now use Fact 8.2.4.

39. F. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then 1 is a singular value of BA but not of AB

40. T, since $A + A^{-1} = A + A^T$ is symmetric.

41. T. The quadratic form $q(x_1, x_2) = \begin{bmatrix} x_1 & 0 & x_2 \end{bmatrix} \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \\ x_2 \end{bmatrix} = ax_1^2 + 2cx_1x_2 + fx_2^2$ is positive definite. The matrix of this quadratic form is $A = \begin{bmatrix} a & c \\ c & f \end{bmatrix}$, and $\det(A) = af - c^2 > 0$ since A is positive definite. Thus $af > c^2$, as claimed.

42. F. Consider the positive definite matrix $A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$.

43. F. Consider the indefinite matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

44. T. By Fact 8.3.2., the continuous function $f(x) = \left\| A \begin{bmatrix} \cos x \\ \sin x \end{bmatrix} \right\|$ has the global maximum 5 and the global minimum 3 $\left(\text{note that the image of the unit circle consists of all vectors of the form } A \begin{bmatrix} \cos x \\ \sin x \end{bmatrix}\right)$. By the intermediate value theorem, $f(c) = 4$ for some c. Let $\vec{u} = \begin{bmatrix} \cos c \\ \sin c \end{bmatrix}$ (draw a sketch!).

45. T, since $\vec{x}^T A^2 \vec{x} = -\vec{x}^T A^T A \vec{x} = -(A\vec{x})^T A\vec{x} = -\|A\vec{x}\|^2 \leq 0$ for all $\vec{x}$.

46. T. If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A^T A$, then $\lambda_1 \lambda_2 \ldots \lambda_n = \det(A^T A) = (\det A)^2$. If $\sigma_1 = \sqrt{\lambda_1}, \ldots, \sigma_n = \sqrt{\lambda_n}$ are the singular values of A, then $\sigma_1 \sigma_2 \ldots \sigma_n = \sqrt{\lambda_1 \lambda_2 \ldots \lambda_n} = |\det A|$, as claimed.

47. F. Note that the columns of S must be unit eigenvectors of A. There are two distinct real eigenvalues, λ_1, λ_2, and for each of them there are two unit eigenvectors, $\pm\vec{v}_1$ (for λ_1) and $\pm\vec{v}_2$ (for λ_2). (Draw a sketch!) Thus there are 8 matrices S, namely $S = [\pm\vec{v}_1 \quad \pm\vec{v}_2]$ and $S = [\pm\vec{v}_2 \quad \pm\vec{v}_1]$

48. T. See the remark following Definition 8.2.1.

49. F. Some eigenvalues of A may be negative.

50. F. Consider the similar matrices $A = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 4 \\ 0 & 3 \end{bmatrix}$. Matrix A has the singular values 0 and 3, while those of B are 0 and 5.

51. T. Let $\vec{v}_1, \vec{v}_2$ be an orthonormal eigenbasis, with $A\vec{v}_1 = \vec{v}_1$ and $A\vec{v}_2 = 2\vec{v}_2$. Consider a nonzero vector $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$; then $A\vec{x} = c_1\vec{v}_1 + 2c_2\vec{v}_2$. If $c_1 = 0$, then $\vec{x} = c_2\vec{v}_2$ and $A\vec{x} = 2c_2\vec{v}_2$ are parallel, and we are all set. Now consider the case when $c_1 \neq 0$. Then the angle between $\vec{x}$ and $A\vec{x}$ is $\arctan(2c_2/c_1) - \arctan(c_2/c_1)$; to see this, subtract the angle between $\vec{v}_1$ and $\vec{x}$ from the angle between $\vec{v}_1$ and $A\vec{x}$ (draw a sketch). Let $m = c_2/c_1$ and use calculus to see that the function $f(m) = \arctan(2m) - \arctan(m)$ assumes its global maximum at $m = \frac{1}{\sqrt{2}}$. The maximal angle between $\vec{x}$ and $A\vec{x}$ is $\arctan(\sqrt{2}) - \arctan(1/\sqrt{2}) < 0.34 < \pi/6$.

52. T. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. By Fact 8.3.2, $\left\| A \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\| = \left\| \begin{bmatrix} a \\ c \end{bmatrix} \right\| = \sqrt{a^2 + c^2} < 5$ (since the length of the semi-major axis of the image of the unit circle is less than 5). Thus $a < 5$ and $c < 5$. Likewise, $b < 5$ and $d < 5$.

53. T. We need to show that each entry $a_{ij} = a_{ji}$ off the diagonal is smaller than some entry on the diagonal. Now $(\vec{e}_i - \vec{e}_j)^T A(\vec{e}_i - \vec{e}_j) = a_{ii} + a_{jj} - 2a_{ij} > 0$, so that $a_{ii} + a_{jj} > 2a_{ij}$. Thus the larger of the diagonal entries a_{ii} and a_{jj} must exceed a_{ij}.