Math S-21b – Summer 2004 – Exam #1 Solutions

1) True or False

a) There exist a 5 by 4 matrix **A** and a 4 by 5 matrix **B** such that **AB** is invertible.

FALSE – **A** is a 5 by 4 matrix, so its rank is at most 4, i.e. the dimension of its image is at most 4. Since the linear transformation given by the matrix **A** is applied last, i.e. **AB**(**x**) = **A**(**Bx**), this means that the image of **AB** is contained in the image of **A**, and the dimension of the image of **AB**, i.e. the rank of **AB**, can therefore be at most 4. However, **AB** is a 5 by 5 matrix and its rank would have to be 5 to be invertible.

b) Let **A** be a 5 by 4 matrix such that dim(ker **A**) = 1. Then for any **b** in $\mathbf{R}^5$, the solution set to **Ax** = **b** is a line parallel to ker **A**.

FALSE – Since **A** is a 5 by 4 matrix, it represents a linear transformation from $\mathbf{R}^4$ to $\mathbf{R}^5$. Since we are given that dim(ker **A**) = 1, we can apply the Rank-Nullity Theorem to conclude that dim(im **A**) = 3. It is therefore quite possible that given any **b** in $\mathbf{R}^5$, it will not lie in the image. Consequently, the system would be inconsistent and there would be no solutions.

c) If **A** is an 8 × 5 matrix, then the kernel of **A** is at least three-dimensional.

FALSE – Such a matrix will represent a linear function from $\mathbf{R}^5$ to $\mathbf{R}^8$. If this function has rank 5, i.e. if its column vectors are linearly independent, then its kernel will consist only of the zero vector in $\mathbf{R}^5$.

d) Consider the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in $\mathbf{R}^m$. Let **A** be a p × m matrix. If the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly dependent, then so are the vectors $\mathbf{A}\mathbf{v}_1, \mathbf{A}\mathbf{v}_2, \ldots, \mathbf{A}\mathbf{v}_n$.

TRUE – If vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ are linearly dependent, then there are constants $c_1, \ldots, c_n$, not all zero, such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = 0$. Apply (multiply by) the matrix **A** on both sides to get: $\mathbf{A}(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n) = c_1\mathbf{A}\mathbf{v}_1 + c_2\mathbf{A}\mathbf{v}_2 + \cdots + c_n\mathbf{A}\mathbf{v}_n = 0$. Thus, there's a nontrivial linear combination of the vectors $\{\mathbf{A}\mathbf{v}_1, \mathbf{A}\mathbf{v}_2, \ldots, \mathbf{A}\mathbf{v}_n\}$ that sums to the zero vector, so they are also a linearly dependent set of vectors.

e) $\text{rank}(\mathbf{A}^2) \le \text{rank}(\mathbf{A})$ for any square matrix **A**. ($\mathbf{A}^2$ denotes the product **AA**.)

TRUE – Use the fact that $\text{im}(\mathbf{A}^2) \subseteq \text{im}(\mathbf{A})$ to conclude that $\dim(\text{im}(\mathbf{A}^2)) \le \dim(\text{im}(\mathbf{A}))$.

f) Let $\mathbf{v} = \begin{bmatrix} 1 \\ 9 \\ 9 \\ 8 \end{bmatrix}$. If **A** and **B** are similar 4 × 4 matrices, and **Av** = **0**, then **Bv** = **0** as well.

FALSE – If **A** and **B** are similar, then there's an invertible 4 × 4 matrix **S** such that $\mathbf{A} = \mathbf{S}^{-1}\mathbf{B}\mathbf{S}$. Since **Av** = **0**, we therefore have that $\mathbf{S}^{-1}\mathbf{B}\mathbf{S}\mathbf{v} = \mathbf{0}$. It follows that **B**(**Sv**) = **0**, so **Sv** is in the kernel of **B**, not **v**.

2) Let **A** be the 4 x 5 matrix **A =** $\begin{bmatrix} 1 & -2 & 1 & 5 & 0 \\ 1 & -2 & -1 & 1 & 0 \\ 3 & -6 & -1 & 7 & -1 \\ -1 & 2 & 0 & -3 & 0 \end{bmatrix}$.

a) Find all solutions of the equation **Ax** = **b**, where **b** = $\begin{bmatrix} 3 \\ 5 \\ 1 \\ -4 \end{bmatrix}$.

Solution: This is solved using row reduction on the augmented matrix:

$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & 5 & 0 & 3 \\ 1 & -2 & -1 & 1 & 0 & 5 \\ 3 & -6 & -1 & 7 & -1 & 1 \\ -1 & 2 & 0 & -3 & 0 & -4 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccccc|c} 1 & -2 & 0 & 3 & 0 & 4 \\ 0 & 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 12 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

There are "leading 1's in the 1st, 3rd, and 5th columns of the reduced augmented matrix, so we'll be able to solve for the corresponding variables and introduce parameters for the others. Thus, we can choose $x_2 = s$ and $x_4 = t$ and solve for the others. So all solutions are of the form:

$$\begin{Bmatrix} x_1 = 4 + 2s - 3t \\ x_2 = s \\ x_3 = -1 - 2t \\ x_4 = t \\ x_5 = 12 \end{Bmatrix} \text{ or in vector form } x = \begin{bmatrix} 4 \\ 0 \\ -1 \\ 0 \\ 12 \end{bmatrix} + s\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}.$$

The solution to this inhomogeneous system is just a translate of ker(**A**), the solution to the homogeneous system.

b) Find a basis for the kernel of **A** and its dimension.

Solution: We solve the homogeneous system **Ax** = **0** by row reduction. This is precisely the same calculation as above only with all zeros on the right side:

$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & 5 & 0 & 0 \\ 1 & -2 & -1 & 1 & 0 & 0 \\ 3 & -6 & -1 & 7 & -1 & 0 \\ -1 & 2 & 0 & -3 & 0 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccccc|c} 1 & -2 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So all solutions are of the form:

$$\begin{Bmatrix} x_1 = 2s - 3t \\ x_2 = s \\ x_3 = -2t \\ x_4 = t \\ x_5 = 0 \end{Bmatrix}, \text{ or in vector form, } \mathbf{x} = s\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}.$$

Thus the kernel is of dimension 2 and these two vectors form a basis for ker(**A**).

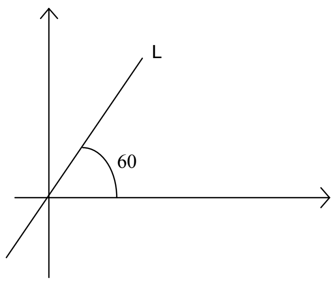
c) Find a basis for the image of **A** and its dimension.

Solution: We've shown that a basis for im(**A**) can be chosen by taking only those column vectors of the original matrix **A** that led to a "leading 1" in rref(**A**). Thus, in this example we may choose the 1st, 3rd, and 5th column vectors of **A**, namely:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \end{bmatrix}, \text{ and } \mathbf{v}_5 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}.$$

Thus the dimension of the image, i.e. the rank of **A**, must be 3.

3) Let T(**x**) = **Ax** be the linear transformation from $\mathbf{R}^2$ to $\mathbf{R}^2$ which first reflects a vector in the line L shown below and dilates the resulting vector by the factor 2.



a) Find the matrix **A**.

b) Find $\mathbf{A}^4\begin{bmatrix}-4\\1\end{bmatrix}$.

[$\mathbf{A}^4$ denotes the product **AAAA**.]

Solution: (a) There are several good ways to do this problem. Perhaps the simplest way is to construct the two columns of the matrix by finding T($\mathbf{e}_1$) and T($\mathbf{e}_2$).

Use the fact that $\text{Proj}_L(\mathbf{x}) = (\mathbf{x}\cdot\mathbf{u})\mathbf{u}$ where **u** is the unit vector $\mathbf{u} = \begin{bmatrix}1/2\\ \sqrt{3}/2\end{bmatrix}$ in the direction of the line L.

This gives $\text{Proj}_L(\mathbf{e}_1) = \frac{1}{2}\begin{bmatrix}1/2\\ \sqrt{3}/2\end{bmatrix} = \begin{bmatrix}1/4\\ \sqrt{3}/4\end{bmatrix}$ and $\text{Proj}_L(\mathbf{e}_2) = \frac{\sqrt{3}}{2}\begin{bmatrix}1/2\\ \sqrt{3}/2\end{bmatrix} = \begin{bmatrix}\sqrt{3}/4\\ 3/4\end{bmatrix}$.

From these we can compute $\text{Ref}_L(\mathbf{e}_1) = 2\,\text{Proj}_L(\mathbf{e}_1) - \mathbf{e}_1 = \begin{bmatrix}1/2\\ \sqrt{3}/2\end{bmatrix} - \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-1/2\\ \sqrt{3}/2\end{bmatrix}$,

so $T(\mathbf{e}_1) = 2\,\text{Ref}_L(\mathbf{e}_1) = \begin{bmatrix}-1\\ \sqrt{3}\end{bmatrix}$.

Likewise, we compute $\text{Ref}_L(\mathbf{e}_2) = 2\,\text{Proj}_L(\mathbf{e}_2) - \mathbf{e}_2 = \begin{bmatrix}\sqrt{3}/2\\ 3/2\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}\sqrt{3}/2\\ 1/2\end{bmatrix}$,

so $T(\mathbf{e}_2) = 2\,\text{Ref}_L(\mathbf{e}_2) = \begin{bmatrix}\sqrt{3}\\ 1\end{bmatrix}$.

This gives us the matrix $\mathbf{A} = \begin{bmatrix}-1 & \sqrt{3}\\ \sqrt{3} & 1\end{bmatrix}$.

(b) Though you certainly can just carry out the matrix multiplications, it's much easier to simply note that this transformation simply flips and stretches by a factor of 2. If you do this four times, you end up in the same direction you started, only stretched by a factor of $2^4 = 16$. That is, $\mathbf{A}^4\begin{bmatrix}-4\\1\end{bmatrix} = 16\begin{bmatrix}-4\\1\end{bmatrix} = \begin{bmatrix}-64\\16\end{bmatrix}$.

4) a) Given the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where $\mathbf{v}_1 = \begin{bmatrix}1\\0\\2\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}0\\2\\1\end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix}0\\1\\1\end{bmatrix}$, show that these three vectors are linearly independent (and are therefore a basis for $\mathbf{R}^3$).

Solution: You can test for linear independence by making a matrix with these three vectors as its columns. The reduced row echelon form of this matrix is the 3×3 identity matrix, has rank 3, and its kernel is $\{\mathbf{0}\}$. Therefore, the three vectors are linearly independent.

b) If $\mathbf{A}\mathbf{v}_1 = \mathbf{v}_2$, $\mathbf{A}\mathbf{v}_2 = \mathbf{v}_1$, and $\mathbf{A}\mathbf{v}_3 = 2\mathbf{v}_3$ (where $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are given above), determine the matrix $\mathbf{A}$.

Solution: There are several good ways to approach this problem. Here's one:

Substituting the given vectors into these three relations, we have:

$$\mathbf{A}\begin{bmatrix}1\\0\\2\end{bmatrix} = \begin{bmatrix}0\\2\\1\end{bmatrix},\ \mathbf{A}\begin{bmatrix}0\\2\\1\end{bmatrix} = \begin{bmatrix}1\\0\\2\end{bmatrix},\ \text{and}\ \mathbf{A}\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\2\\2\end{bmatrix}.$$

These three can be rolled into one matrix statement, namely $\mathbf{A}\begin{bmatrix}1&0&0\\0&2&1\\2&1&1\end{bmatrix} = \begin{bmatrix}0&1&0\\2&0&2\\1&2&2\end{bmatrix}$.

Calling these two matrices $\mathbf{B}$ and $\mathbf{C}$, we can write this as $\mathbf{AB} = \mathbf{C}$. If we further note (from part (a)) that $\mathbf{B}$ is invertible, we can solve for $\mathbf{A}$ as $\mathbf{A} = \mathbf{CB}^{-1}$. We calculate this inverse matrix using row reduction or a calculator and do the matrix multiplication thus:

$$\mathbf{A} = \begin{bmatrix}0&1&0\\2&0&2\\1&2&2\end{bmatrix}\begin{bmatrix}1&0&0\\2&1&-1\\-4&-1&2\end{bmatrix} = \begin{bmatrix}2&1&-1\\-6&-2&4\\-3&0&2\end{bmatrix}.$$

Alternatively, you could find the matrix of this transformation relative to the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, then conjugate by the change of basis matrix to find $\mathbf{A}$, the matrix relative to the standard basis.

c) Find a vector $\mathbf{x}$ such that $\mathbf{Ax} = -\mathbf{x}$.

Solution 1: If you simply treat this as a system of linear equations, you get:

$$\begin{Bmatrix} 2x_1 + x_2 - x_3 = -x_1 \\ -6x_1 - 2x_2 + 4x_3 = -x_2 \\ -3x_1 \quad + 2x_3 = -x_3 \end{Bmatrix} \Rightarrow \begin{Bmatrix} 3x_1 + x_2 - x_3 = 0 \\ -6x_1 - x_2 + 4x_3 = 0 \\ -3x_1 \quad + 3x_3 = 0 \end{Bmatrix} \Rightarrow \left[\begin{array}{ccc|c}3&1&-1&0\\-6&-1&4&0\\-3&0&3&0\end{array}\right] \Rightarrow \left[\begin{array}{ccc|c}1&0&-1&0\\0&1&2&0\\0&0&0&0\end{array}\right]$$

The solutions are therefore of the form: $\begin{Bmatrix} x_1 = t \\ x_2 = -2t \\ x_3 = t \end{Bmatrix}$, or $\mathbf{x} = t\begin{bmatrix}1\\-2\\1\end{bmatrix}$. In particular, $\begin{bmatrix}1\\-2\\1\end{bmatrix}$ is a solution.

Solution 2: We can do the same solution as above with matrices: $\mathbf{Ax} = -\mathbf{x} = -\mathbf{Ix}$. This gives us $\mathbf{Ax} + \mathbf{Ix} = \mathbf{0}$ or, more simply, $(\mathbf{A} + \mathbf{I})\mathbf{x} = \mathbf{0}$, and this is solved by row reduction as above.

Solution 3: If you note that $\mathbf{A}\mathbf{v}_1 = \mathbf{v}_2$ and $\mathbf{A}\mathbf{v}_2 = \mathbf{v}_1$, you can observe that:

$$\mathbf{A}(\mathbf{v}_1 - \mathbf{v}_2) = \mathbf{A}\mathbf{v}_1 - \mathbf{A}\mathbf{v}_2 = \mathbf{v}_2 - \mathbf{v}_1 = -(\mathbf{v}_1 - \mathbf{v}_2).$$

Therefore, the vector $(\mathbf{v}_1 - \mathbf{v}_2)$ is a solution. (It's the same as given above in Solution 1.)