Math S-21b – Summer 2004 – Exam #2 Solutions

1) TRUE/FALSE

a) A linear transformation T: $\mathbf{R}^n \to \mathbf{R}^n$ is an orthogonal transformation if and only if $T(\mathbf{x})\bullet T(\mathbf{y}) = \mathbf{x}\bullet\mathbf{y}$ for all vectors **x** and **y** in $\mathbf{R}^n$.

TRUE – You can easily verify that if this condition holds, then T preserves norm. Hence T will be orthogonal. That an orthogonal transformation preserves the dot product was proven in class.

b) If T is a linear transformation from $\mathbf{R}^n$ to $\mathbf{R}^n$ which sends orthogonal vectors to orthogonal vectors, then T is an orthogonal transformation.

FALSE – For example, a dilation that scales by anything other than 1 or –1 will preserve orthogonality but not norm, and will therefore not be an orthogonal transformation.

c) Let **A** be a square matrix with exactly one entry 1 in each row and in each column, the other entries being zero. Then **A** is an orthogonal matrix.

TRUE – Every column is a unit vector and they are mutually orthogonal.

d) If **A** is an $n \times n$ matrix, then $\det(2\mathbf{A}) = 2(\det \mathbf{A})$.

FALSE – The determinant is linear in any one column or row, but it's not a linear function. In fact, $\det(2\mathbf{A}) = 2^n(\det \mathbf{A})$ for an $n \times n$ matrix.

e) There is a real 4×4 matrix **A** such that $\det(\mathbf{A}^T\mathbf{A}^2\mathbf{A}^T) = -16$. (Here $\mathbf{A}^T$ denotes the transpose of **A**.)

FALSE - $\det(\mathbf{A}^T\mathbf{A}^2\mathbf{A}^T) = \det(\mathbf{A}^T)\det(\mathbf{A}^2)\det(\mathbf{A}^T) = \det(\mathbf{A})[\det(\mathbf{A})]^2\det(\mathbf{A}) = [\det(\mathbf{A})]^4 = -16$. This is clearly impossible for a matrix **A** (with real entries) since the determinant will be real and there is no real number which, when raised to the 4th power, will give a negative number.

f) If **A** is a 4×4 matrix with $\mathbf{A}^2 = \mathbf{A}$, then the only possible eigenvalues of **A** are 0 and 1.

TRUE – Suppose **v** is an eigenvector of **A** with eigenvalue λ. Then $\mathbf{Av} = \lambda\mathbf{v}$ and $\mathbf{A}^2\mathbf{v} = \mathbf{A}(\mathbf{Av}) = \mathbf{A}(\lambda\mathbf{v}) = \lambda\mathbf{Av} = \lambda^2\mathbf{v}$. But $\mathbf{A}^2\mathbf{v} = \mathbf{Av}$, so $\lambda^2\mathbf{v} = \lambda\mathbf{v}$. Therefore $(\lambda^2 - \lambda)\mathbf{v} = \lambda(\lambda - 1)\mathbf{v} = \mathbf{0}$. Since **v** is not the zero vector, it follows that $\lambda(\lambda - 1) = 0$, so either $\lambda = 0$ or $\lambda = 1$.

2) Consider the linear transformation T: $\mathbf{R}^{2\times2} \to \mathbf{R}^{2\times2}$ given by $T(\mathbf{A}) = \mathbf{A}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\mathbf{A}$.

a) Calculate $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$ and <u>find the matrix **M**</u> of T with respect to the standard basis

$$\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\} \text{ of } \mathbf{R}^{2\times2}.$$

Solution: Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and calculate

$$T(\mathbf{A}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} b & -a \\ d & -c \end{bmatrix} - \begin{bmatrix} -c & -d \\ a & b \end{bmatrix} = \begin{bmatrix} (b+c) & (d-a) \\ (d-a) & -(b+c) \end{bmatrix}.$$

Since the coordinate vector of **A** relative to this standard basis is $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ and T(**A**) has coordinates

$$\begin{bmatrix} b+c \\ d-a \\ d-a \\ -b-c \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}, \text{ it follows that the matrix } \mathbf{M} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix}.$$

b) Find bases of the kernel and image of T.

Solution: You can find these bases using the matrix **M** and row reduction, or you can simply note that if T(**A**) = **0**, then $b + c = 0$ and $d - a = 0$, so $c = -b$ and $d = a$, and **A** must be of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Therefore, the kernel has the basis $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\}$.

Similarly, anything in the image is of the form $\begin{bmatrix} b+c & d-a \\ d-a & -(b+c) \end{bmatrix} = (b+c)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + (d-a)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, hence the image has the basis $\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}$.

c) Is T invertible? Explain.

Solution: T is not invertible because the kernel is nontrivial.

3) Let V be the subspace of $\mathbf{R}^4$ spanned by the two vectors $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 4 \\ 0 \\ 1 \end{bmatrix}$.

a) Find an orthonormal basis $\{\mathbf{w}_1, \mathbf{w}_2\}$ for V using the Gram-Schmidt method.

Solution: $\|\mathbf{v}_1\| = 3$, so $\mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

If we subtract the projection of $\mathbf{v}_2$ in the direction, we get $\mathbf{v}_2 - (\mathbf{v}_2 \cdot \mathbf{w}_1)\mathbf{w}_1 = \begin{bmatrix} 1 \\ 4 \\ 0 \\ 1 \end{bmatrix} - 4\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

Normalizing this, we get $\mathbf{w}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. So the orthonormal basis is $\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$.

b) Find the area of the parallelogram spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$.

Solution: Let $\mathbf{A} = \begin{bmatrix} \uparrow & \uparrow \\ \mathbf{v}_1 & \mathbf{v}_2 \\ \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3 & 4 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then $\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 9 & 12 \\ 12 & 18 \end{bmatrix}$ and Area = $\sqrt{\det(\mathbf{A}^T\mathbf{A})} = \sqrt{18} = 3\sqrt{2}$.

c) Find the matrix **P** for orthogonal projection onto the subspace V.

Solution: Using the <u>orthonormal</u> basis for V found in part (a), we let $\mathbf{B} = \begin{bmatrix} \uparrow & \uparrow \\ \mathbf{w}_1 & \mathbf{w}_2 \\ \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 0 & 1/\sqrt{2} \\ 1 & 0 \\ 0 & 0 \\ 0 & 1/\sqrt{2} \end{bmatrix}$.

We've previously shown that the matrix for orthogonal projection onto V is:

$$\mathbf{P} = \mathbf{B}\mathbf{B}^T = \begin{bmatrix} 0 & 1/\sqrt{2} \\ 1 & 0 \\ 0 & 0 \\ 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{bmatrix}.$$

d) Find the matrix **R** for reflection through the subspace V.

Solution: We've previously shown that the matrix for reflection through the subspace V is given by:

$$\mathbf{R} = 2\mathbf{P} - \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

4) a) Find the linear function $f(x) = mx + b$ that best fits the 4 points (-1, 0), (0,0), (1,1), (2,0) in the sense of least squares.

Solution: If we substitute each of these four points into the equation $y = mx + b$, we get the four inconsistent linear equations $\begin{Bmatrix} -m+b=0 \\ b=0 \\ m+b=1 \\ 2m+b=0 \end{Bmatrix}$. In matrix form, this is $\mathbf{Ac} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \mathbf{y}$.

The least-squares solution is found by solving the <u>normal equation</u> $\mathbf{A}^{\mathrm{T}}\mathbf{Ac} = \mathbf{A}^{\mathrm{T}}\mathbf{y}$.

This gives us $\begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, which we solve to get $m = 1/10$ and $b = 1/5$.

b) Find the quadratic function $f(x) = a + bx + cx^2$ that best fits the 4 points (-1, 0), (0,0), (1,1), (2,0) in the sense of least squares.

Solution: This is similar to the above calculation, except that when we substitute the four points into the equation $y = a + bx + cx^2$, we get the four inconsistent linear equations $\begin{Bmatrix} a-b+c=0 \\ a=0 \\ a+b+c=1 \\ a+2b+4c=0 \end{Bmatrix}$.

In matrix form, this is $\mathbf{Ac} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \mathbf{y}$.

The least-squares solution is again found by solving the <u>normal equation</u> $\mathbf{A}^{\mathrm{T}}\mathbf{Ac} = \mathbf{A}^{\mathrm{T}}\mathbf{y}$.

This gives us $\begin{bmatrix} 4 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, which we solve to get $a = 9/20$, $b = 7/20$, and $c = -1/4$.