Math S-21b – Summer 2004 – Practice Exam #1 Solutions

(1) True or False.

a) If T_1, T_2 are two linear transformations from $\mathbf{R}^3$ to $\mathbf{R}^3$ such that $\ker(T_1) = \ker(T_2)$ and $\text{image}(T_1) = \text{image}(T_2)$, then $T_1 = T_2$.

FALSE – Consider a projection onto a two-dimensional subspace vs. the same projection followed by a rotation in that subspace. In both cases, the kernel is the same (orthogonal complement) subspace. Also the image, in both cases, is the plane onto which we're projecting. However, in the first case we don't rotate and the second case we do, so these two transformations are not the same.

b) The kernel of rref(**A**) is the same as the kernel of **A**.

TRUE - The reason why this is true is because row reduction of the augmented **A** matrix yields the augmented rref(**A**) matrix. Thus, the solutions to the original system and the solutions to the reduced system are exactly the same.

c) The image of rref(**A**) is the same as the image of **A**.

FALSE - It's pretty simple to see that this is false just by considering a few simple examples. The original columns of the matrix **A** can be complicated, but the columns of rref(**A**) are either elementary vectors or scalar multiples of them.

d) If **A** and **B** are n x n matrices such that the kernel of **A** is contained in the image of **B**, then the matrix **AB** cannot be invertible.

FALSE - As a counterexample, chose both **A** and **B** to be invertible matrices. The kernel of **A** will be {**0**}, the image of **B** will be all of $\mathbf{R}^n$ (which contains **0**), and the product **AB** will be invertible.

e) Let **A** and **B** be n x n matrices, with **AB** = **BA**. Then $\mathbf{A}^3\mathbf{B} = \mathbf{B}\mathbf{A}^3$.

TRUE - This one's pretty obvious. If you can reverse the order of the factors, then you can get from the left-hand side to the right-hand side in three steps.

f) If **A** is an n x n matrix, $\mathbf{A}^2 = \mathbf{A}$, and rank(**A**) = n, then $\mathbf{A} = \mathbf{I}_n$.

TRUE – If rank(**A**) = n, then **A** is invertible. If we multiply both sides of the equation $\mathbf{A}^2 = \mathbf{A}$ by $\mathbf{A}^{-1}$ we get that $\mathbf{A} = \mathbf{I}_n$.

2) Short answer questions:

a) If $\mathbf{B} = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 4 & 2 \\ 1 & 0 \end{bmatrix}$ and **AB = C**, what is **A**?

Solution: Just note that if we multiply both sides on the right by $\mathbf{B}^{-1}$, you get

$$\mathbf{A} = \mathbf{C}\mathbf{B}^{-1} = \begin{bmatrix} 4 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}$$

b) Let $\mathbf{A} = \begin{bmatrix} 5 & -12 \\ 12 & 5 \end{bmatrix}$.

Describe <u>briefly</u>, in geometric terms, the linear transformation represented by this matrix.

Solution: This matrix is in the form $\mathbf{A} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, so it must represent a rotation-dilation. More specifically,

$\mathbf{A} = \begin{bmatrix} 5 & -12 \\ 12 & 5 \end{bmatrix} = 13\begin{bmatrix} 5/13 & -12/13 \\ 12/13 & 5/13 \end{bmatrix}$, so it represents the composition of a rotation through the angle $\theta = \tan^{-1}(12/5)$ and a dilation by a factor of 13.

c) For which choices of the constant k is the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & k \\ 1 & 4 & k^2 \end{bmatrix}$ not invertible? Explain briefly.

Solution: The matrix will be invertible if it has full rank = 3. After a little row reduction, you're left with

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2-k \\ 0 & 1 & k-1 \\ 0 & 0 & k^2-3k+2 \end{bmatrix}$$

This will fail to have rank 3 only when $k^2 - 3k + 2 = (k-1)(k-2) = 0$, so $k = 1$ or $k = 2$.

d) Find a basis for the subspace of $\mathbf{R}^4$ that consists of all vectors orthogonal to both $\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$.

Solution: Suppose a vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ was orthogonal (perpendicular) to both of these vectors. Then the dot product with each would be zero. This translates into the two homogeneous linear equations: $\left\{ \begin{array}{r} x_1 - x_3 + x_4 = 0 \\ x_2 + 2x_3 + 3x_4 = 0 \end{array} \right\}$.

Using row reduction to solve these. We find that this system is already in reduced row echelon form: $\left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{array}\right]$.

The solutions are therefore of the form: $\left\{ \begin{array}{l} x_1 = s - t \\ x_2 = -2s - 3t \\ x_3 = s \\ x_4 = t \end{array} \right\}$ or $\mathbf{x} = s\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \end{bmatrix}$. So the basis is $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

3) Let **A** be the 3 x 5 matrix **A** = $\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ -1 & 3 & -5 & 2 & 8 \\ 2 & 3 & 1 & 2 & 5 \end{bmatrix}$.

a) <u>Find a basis for the kernel</u> of **A** and <u>its dimension</u>, i.e. the nullity.

Solution: We must solve the equation **Ax** = **0**. This is a homogeneous system of 3 equations in 5 unknowns and is best solved using row reduction.

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 3 & -5 & 2 & 8 & 0 \\ 2 & 3 & 1 & 2 & 5 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{array}\right]$$

There are "leading 1's" in the 1st, 2nd, and 4th columns of the reduced augmented matrix, so we'll be able to solve for the corresponding variables and introduce parameters for the others. Thus, we can choose $x_3 = s$ and $x_5 = t$ and solve for the others to get:

$$\left\{ \begin{array}{l} x_1 = -2s + t \\ x_2 = s - t \\ x_3 = s \\ x_4 = -2t \\ x_5 = t \end{array} \right\} \text{ or } \mathbf{x} = s\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

Thus the nullity(**A**) = dim(ker **A**) = 2, and these two vectors form a basis for ker(**A**).

b) <u>Find a basis for the image</u> of **A** and <u>its dimension</u>, i.e. the rank.

Solution: We've shown that a basis for im(**A**) can be chosen by taking only those column vectors of the original matrix **A** that led to a "leading 1" in rref(**A**). Thus, in this example we may chose the 1st, 2nd, and 4th columns vectors of **A**, namely:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \text{ and } \mathbf{v}_4 = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}.$$

Thus the dimension of the image, i.e. the <u>rank</u> of **A**, must be 3. However, since these three vectors span all of $\mathbf{R}^3$, you can actually choose any basis for $\mathbf{R}^3$ and it will serve as a basis for im(**A**).

c) Find all solutions of the equation $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 3 \\ 5 \\ 11 \end{bmatrix}$.

Solution: This solution proceeds in much the same way as in finding the kernel, only the system is no longer homogeneous and thus the matrix must be augmented with other than a column of zeros. Specifically:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 3 \\ -1 & 3 & -5 & 2 & 8 & 5 \\ 2 & 3 & 1 & 2 & 5 & 11 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -1 & 2 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 2 \end{array}\right]$$

So all solutions are of the form:

$$\left\{\begin{array}{l} x_1 = 2 - 2s + t \\ x_2 = 1 + s - t \\ x_3 = s \\ x_4 = 2 - 2t \\ x_5 = t \end{array}\right\} \text{ or } \mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

Thus, the solution to the inhomogeneous system is just a translate of ker(**A**), the solution to the homogeneous system.

4) Let $\mathcal{B}$ be the basis of $\mathbf{R}^3$ consisting of the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Let T: $\mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation such that $T(\mathbf{v}_1) = \mathbf{v}_2$, $T(\mathbf{v}_2) = \mathbf{v}_1$, and $T(\mathbf{v}_3) = -\mathbf{v}_3$.

a) Find the coordinates of the vector $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ relative to the basis $\mathcal{B}$.

Solution: If we let $\mathbf{S} = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ be the change of basis matrix whose columns are these three vectors, then the coordinates of $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ relative to the basis $\mathcal{B}$ is given by $\mathbf{S}^{-1}\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix}$.

b) Find the matrix **B** of T with respect to the basis $\mathcal{B}$.

Solution: We get **B** simply by interpreting the given rules describing what the image of the basis vectors are expressed in terms of that basis. That is,

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

c) Find the matrix **A** of T with respect to the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathbf{R}^3$.

Solution: We can use the fact that $\mathbf{S}^{-1}\mathbf{AS} = \mathbf{B}$, to get that

$$\mathbf{A} = \mathbf{SBS}^{-1} = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -8 & -4 \\ 1 & -3 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$