Math S-21b – Summer 2004 – Solutions to Practice Exam #2

1) TRUE/FALSE (circle one)

a) Orthogonal projection onto a subspace is an orthogonal transformation.

FALSE – Orthogonal transformations must preserve length. Projections do not.

b) If **A** is any square matrix, then det $(-\mathbf{A}) = -\det(\mathbf{A})$.

FALSE – The determinant is linear in any one column or row, but it's not a linear function of the matrix. It is true, however, that $\det(-\mathbf{A}) = (-1)^n \det(\mathbf{A})$ for an $n \times n$ matrix **A**.

c) If **A** is an invertible matrix that is <u>similar</u> to its own inverse $\mathbf{A}^{-1}$, then $\det(\mathbf{A})$ must be either +1 or –1.

TRUE – The determinant of similar matrices is the same, and $\det(\mathbf{A}^{-1}) = (\det \mathbf{A})^{-1}$, so $\det(\mathbf{A}) = (\det \mathbf{A})^{-1}$. Hence $[\det(\mathbf{A})]^2 = 1$. It follows, therefore, that $\det(\mathbf{A})$ must equal either +1 or –1.

d) Let **A** be an orthogonal matrix. Then $\det(\mathbf{A}) = 1$.

FALSE – Either $\det(\mathbf{A}) = +1$ or $\det(\mathbf{A}) = -1$ for an orthogonal matrix.

e) If **S** and **A** are orthogonal $n \times n$ matrices, then the matrix $\mathbf{S}^{-1}\mathbf{AS}$ is orthogonal as well.

TRUE – If **S** and **A** are orthogonal matrices, then they preserve norm. $\mathbf{S}^{-1}$ will also preserve norm. Therefore the composition given by the matrix $\mathbf{S}^{-1}\mathbf{AS}$ will preserve norm and will thus be orthogonal.

f) It is always the case that $\det(\mathbf{A}^T\mathbf{A}) \geq 0$.

TRUE – In fact, if $\mathbf{A} = \begin{bmatrix} \uparrow & & \uparrow \\ \mathbf{v}_1 & \cdots & \mathbf{v}_k \\ \downarrow & & \downarrow \end{bmatrix}$, then the k-volume of the parallelepiped determined by $\{v_1,\ldots,v_k\}$ will be given by $\sqrt{\det(\mathbf{A}^T\mathbf{A})}$.

g) If λ is an eigenvalue of **A** with eigenvector **v**, μ is a distinct eigenvalue with eigenvector **w**, then **v** + **w** is also an eigenvector of **A**.

FALSE – What is true is that $\mathbf{Av} = \lambda\mathbf{v}$ and $\mathbf{Aw} = \mu\mathbf{w}$, so $\mathbf{A}(\mathbf{v} + \mathbf{w}) = \lambda\mathbf{v} + \mu\mathbf{w}$, and this will be a scalar multiple of **v** + **w** only if $\lambda = \mu$.

h) If **A** is an n × n matrix, then **A** and $\mathbf{A}^T$ have the same eigenvalues.

TRUE – The characteristic polynomial of **A** is $\det(\lambda\mathbf{I} - \mathbf{A})$ and since $(\lambda\mathbf{I} - \mathbf{A})^T = (\lambda\mathbf{I} - \mathbf{A}^T)$, it follows that $\det(\lambda\mathbf{I} - \mathbf{A}^T) = \det(\lambda\mathbf{I} - \mathbf{A})^T = \det(\lambda\mathbf{I} - \mathbf{A})$. So **A** and $\mathbf{A}^T$ have the same characteristic polynomials. Since their eigenvalues are just the roots of their characteristic polynomials (counting algebraic multiplicity), they will therefore also have the same eigenvalues.

2) Let P_2 is the linear space consisting of all polynomials of degree ≤ 2, and let $T: P_2 \to P_2$ be defined by

$$T(f) = f'' - 2f' + 3f\ .$$

a) Find the matrix of this linear transformation relative to the basis $\{1, t, t^2\}$.

Solution: Let $f(t) = a + bt + ct^2$. Then $f'(t) = b + 2ct$ and $f''(t) = 2c$. Therefore:

$[T(f)](t) = f''(t) - 2f'(t) + 3f(t) = (2c) - 2(b + 2ct) + 3(a + bt + ct^2) = (3a - 2b + 2c) + (3b - 4c)t + 3ct^2$**.**

This says that, relative to the basis $\{1, t, t^2\}$,

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \to \begin{bmatrix} 3a - 2b + 2c \\ 3b - 4c \\ 3c \end{bmatrix} = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Therefore, the matrix of this linear transformation relative to the basis $\{1, t, t^2\}$ is $\begin{bmatrix} 3 & -2 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 3 \end{bmatrix}$.

b) Find the determinant of this linear transformation.

Solution: From part (a) and the fact that the determinant does not depend on the basis, we easily see that det(T) = 27.

c) Is this linear transformation invertible, i.e. an isomorphism? Briefly explain.

Solution: Since det(T) ≠ 0, this linear transformation is invertible.

d) Using the above information, find a quadratic polynomial $f(t)$ such that $f'' - 2f' + 3f = 3t^2$.

Solution: We want to find a function *f* such that $T(f) = 3t^2$. Using the basis above and the matrix of T relative to this basis, this translates into solving $\begin{bmatrix} 3 & -2 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$. This can be solved using row reduction or matrix inversion to get $a = \frac{2}{9}, b = \frac{4}{3}, c = 1$. So the function that solves this inhomogeneous equation is $f(t) = \frac{2}{9} + \frac{4}{3}t + t^2$.

3) Consider the following <u>inconsistent</u> system of linear equations: $\left\{ \begin{array}{c} 2x + y = 2 \\ x - y = 1 \\ x + 2y = -2 \end{array} \right\}$.

a) Find the least-squares solution for this linear system. [*Note*: This is <u>not</u> a data-fitting problem.]

Solution: This inconsistent system is of the form **Ax** = **b**, where $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 1 & 2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$.

The least-squares solution is the solution of the normal equation $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$ or $\begin{bmatrix} 6 & 3 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$.

Solving this gives the point $(x, y) = (1, -1)$.

b) Each equation in the given system represents a line in $\mathbf{R}^2$. Describe in words and/or pictures the relationship between these three lines and the point you found in part a).

Solution: If you draw these three lines, you'll find that the least-squares solution is at the center of the triangle formed by these three lines.

4) We are given three vectors in $\mathbf{R}^4$: $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$.

a) Find the area of the parallelogram determined by the vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$.

Solution: Let $\mathbf{A}$ be the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \\ -1 & 0 \end{bmatrix}$. We calculate $\mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$, so the area will be given by $\sqrt{\det(\mathbf{A}^{\mathrm{T}}\mathbf{A})} = \sqrt{4} = 2$.

b) Construct an <u>orthonormal basis</u> for the two-dimensional subspace of $\mathbf{R}^4$ spanned by $\{\mathbf{v}_1, \mathbf{v}_2\}$. Call the vectors of this orthonormal basis $\mathbf{w}_1$ and $\mathbf{w}_2$.

Solution: This is a simple Gram-Schmidt orthogonalization problem. We have:

$\|\mathbf{v}_1\| = 2$, so we take $\mathbf{w}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$. We then calculate $\mathbf{v}_2 - (\mathbf{v}_2 \bullet \mathbf{w}_1)\mathbf{w}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{w}_2$, since it's already a unit vector.

c) Find the orthogonal projection of $\mathbf{v}_3$ in the subspace spanned by the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

Solution: We calculate $\mathrm{Proj}_V(\mathbf{v}_3) = (\mathbf{v}_3 \bullet \mathbf{w}_1)\mathbf{w}_1 + (\mathbf{v}_3 \bullet \mathbf{w}_2)\mathbf{w}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ -1 \end{bmatrix}$.

d) If we let $\mathbf{B} = \begin{bmatrix} \uparrow & \uparrow \\ \mathbf{w}_1 & \mathbf{w}_2 \\ \downarrow & \downarrow \end{bmatrix}$ where $\{\mathbf{w}_1, \mathbf{w}_2\}$ is the orthonormal basis found in part b, what are the values of $\det(\mathbf{B}\mathbf{B}^{\mathrm{T}})$ and $\det(\mathbf{B}^{\mathrm{T}}\mathbf{B})$?
[You don't need to know what $\mathbf{w}_1$ and $\mathbf{w}_2$ are to calculate these two numbers.]

Solution: First, $\mathbf{B}\mathbf{B}^{\mathrm{T}}$ is the matrix for <u>orthogonal projection</u> onto the subspace spanned by $\{\mathbf{w}_1, \mathbf{w}_2\}$. It follows that $\det(\mathbf{B}\mathbf{B}^{\mathrm{T}}) = 0$.

Next, $\mathbf{B}^{\mathrm{T}}\mathbf{B} = \begin{bmatrix} \leftarrow & \mathbf{w}_1 & \rightarrow \\ \leftarrow & \mathbf{w}_2 & \rightarrow \end{bmatrix}\begin{bmatrix} \uparrow & \uparrow \\ \mathbf{w}_1 & \mathbf{w}_2 \\ \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so $\det(\mathbf{B}^{\mathrm{T}}\mathbf{B}) = 1$.