Math S-21b Practice Final Exam – Summer 2004

The topics, difficulty level, and number of questions may be different on the actual exam.

1) **True/False** (circle one)

(a) If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices such that $\mathbf{AB} = \mathbf{BA}$, and if $\mathbf{v}$ is an eigenvector of $\mathbf{A}$, then $\mathbf{Bv}$ is also an eigenvector of $\mathbf{A}$.	TRUE or FALSE
(b) If $\mathbf{A}$ is not a square matrix, then $\mathbf{AA}^T$ is not invertible.	TRUE or FALSE
(c) If $\mathbf{A}$ is a real 5×4 matrix, then $\mathbf{AA}^T$ is positive definite.	TRUE or FALSE
(d) If the columns of an $m \times n$ matrix $\mathbf{A}$ are linearly independent, then the columns of its transpose $\mathbf{A}^T$ will be linearly independent as well.	TRUE or FALSE
(e) For all real numbers c the matrix $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ c & 0 & 1 \end{bmatrix}$ is invertible.	TRUE or FALSE
(f) If $\mathbf{v}$ is a unit (column) vector in $\mathbf{R}^3$, then the matrix $\mathbf{vv}^T$ is diagonalizable.	TRUE or FALSE
(g) If two matrices have the same characteristic polynomial, then they have the same rank.	TRUE or FALSE
(h) Any symmetric 2×2 matrix has two distinct eigenvalues.	TRUE or FALSE

2) Short answer questions:

a) Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$. Calculate the area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$.

b) Find the matrix representing the linear transformation from $\mathbf{R}^2$ to $\mathbf{R}^2$ that is reflection in the line spanned by the vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

c) Find a matrix with eigenvalues equal to 2,3,5,7.

d) Consider the vector space V consisting of all 2×2 matrices for which the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector. Find a basis for this space, and determine its dimension.

3) Consider a linear transformation $\mathrm{T}: \mathbf{R}^2 \to \mathbf{R}^2$. Suppose the matrix of T with respect to the basis $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$ is $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Find the matrix of T with respect to the basis $\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\}$.

4) Let $\mathbf{A}$ be a real $n \times n$ matrix such that $\mathbf{A}^2 = -\mathbf{I}_n$.

a) Show that $\mathbf{A}$ is invertible.
b) Show that n must be even.
c) Show that $\mathbf{A}$ has no real eigenvalues.

5) Consider the quadratic form $q(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 + 2x_1x_3$.

a) Find a symmetric matrix **A** such that $q(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x}$ for all $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in $\mathbf{R}^3$.

b) Find all the eigenvalues of **A** and their algebraic and geometric multiplicities.

c) Is **A** positive definite? Briefly justify your answer.

d) Find an orthonormal eigenbasis for **A**.

6) A rabbit population and a wolf population are modeled by the equations

$$r(t+1) = 5r(t) - 2w(t)$$
$$w(t+1) = r(t) + 2w(t)$$

The initial populations are $r(0)=300$ and $w(0)=200$.

a) Find formulas for $r(t)$ and $w(t)$.

b) In the long run, what will be the proportion of rabbits to wolves? Explain.

7) Let $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -b & -c \end{bmatrix}$ where b and c are real numbers.

Consider the continuous dynamical system $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$.

a) What inequality or inequalities involving b and c ensure that the solutions to the system will consist of trajectories spiraling inwards toward the origin?

b) Solve this continuous dynamical system in the case where $b = 4$, $c = 5$, and $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(Your answer should be a closed formula for $\mathbf{x}(t)$.)

c) Sketch the phase portrait for this continuous dynamical system.

8) Consider the system $\left\{ \begin{array}{l} \frac{dx}{dt} = x - y \\ \frac{dy}{dt} = x^2 - y \end{array} \right\}$.

a) Perform the qualitative phase plane analysis for this system (i.e., find the null clines, equilibrium points, and general directions). Carry this out in the whole xy-plane (not just the first quadrant).

b) List the equilibrium points of the system above, and determine their stability. That is, linearize the system at each equilibrium and do the eigenvalue-eigenvector analysis. Give a rough sketch of some solutions, particularly in the vicinity of the equilibria.

BONUS QUESTION

Consider the two subspaces V_1 and V_2 of $\mathbf{R}^4$, where $V_1 = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix} \right\}$ and $V_2 = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Find a basis for the intersection $V_1 \cap V_2$. [Note: The intersection of two subspaces is also a subspace.]