MATH S-21b PRACTICE FINAL EXAM SOLUTIONS – 2004: SOLUTIONS

100 points total. Calculators are allowed, but explain your calculations.
Use the reverse sides of each page for additional work. Vectors and matrices are indicated in **bold**.

1) (2 pts each) **True/False** (circle one)

(a) If **A** and **B** are $n \times n$ matrices such that **AB** = **BA**, and if **v** is an eigenvector of **A**, then **Bv** is also an eigenvector of **A**.	TRUE or FALSE
(b) If **A** is not a square matrix, then $\mathbf{AA}^T$ is not invertible.	TRUE or FALSE
(c) If **A** is a real 5×4 matrix, then $\mathbf{AA}^T$ is positive definite.	TRUE or FALSE
(d) If the columns of an $m \times n$ matrix **A** are linearly independent, then the columns of its transpose $\mathbf{A}^T$ will be linearly independent as well.	TRUE or FALSE
(e) For all real numbers c the matrix $\begin{bmatrix}1&1&0\\0&1&1\\c&0&1\end{bmatrix}$ is invertible.	TRUE or FALSE
(f) If **v** is a unit (column) vector in $\mathbf{R}^3$, then the matrix $\mathbf{vv}^T$ is diagonalizable.	TRUE or FALSE
(g) If two matrices have the same characteristic polynomial, then they have the same rank.	TRUE or FALSE
(h) Any symmetric 2×2 matrix has two distinct eigenvalues.	TRUE or FALSE

(a) $A\vec v = \lambda \vec v$. $A(B\vec v) = AB\vec v = BA\vec v = B(\lambda\vec v) = \lambda(B\vec v) \Rightarrow B\vec v$ an e-vector of A.

(b) Example: $A = [1\ \ 2]$, $AA^T = [1\ \ 2]\begin{bmatrix}1\\2\end{bmatrix} = [5]$. (invertible 1×1 MATRIX)

(c) Example: $A = [\vec v_1\ \vec v_2\ \vec v_3\ \vec v_4]$ w/ ON columns $\Rightarrow AA^T$ is ORTH. PROJ. onto Span$\{\vec v_1, \dots, \vec v_4\}$
e-vals are $\{1,1,1,1,0\}$ ⇒ not pos. definite.

(d) Example: $\begin{bmatrix}1&0\\0&1\\1&1\end{bmatrix} = A$ has lin indep. columns, $A^T = \begin{bmatrix}1&0&1\\0&1&1\end{bmatrix}$ does not have lin. indep. columns.
Note: rank(A) = rank(A^T) always, so same no. of lin. indep cols. in all cases.

(e) $\det = 1[1] + c[1] = 1 + c = 0$ when $c = -1$. ⇒ not invertible.

(f) The MATRIX $\vec v \vec v^T$ will, in this case, give ORTH. Projection onto Span$\{\vec v\}$ ⇒ DIAGONALIZABLE.
Also, $\vec v \vec v^T$ is symmetric, hence orthogonally diagonalizable.

(g) Example: $Z = \begin{bmatrix}0&0\\0&0\end{bmatrix}$ vs. $A = \begin{bmatrix}0&1\\0&0\end{bmatrix}$ $|\lambda I - Z| = \begin{vmatrix}\lambda&0\\0&\lambda\end{vmatrix} = \lambda^2$ ⇒ repeated evals $\lambda = 0$.
$|\lambda I - A| = \begin{vmatrix}\lambda&-1\\0&\lambda\end{vmatrix} = \lambda^2$
$\text{Ker}(Z) = \mathbb{R}^2 \Rightarrow$ rank $= 0$
$\text{Ker}(A) = \text{Span}\{\vec e_1\} \Rightarrow$ rank $= 1$.

(h) Example: $I = \begin{bmatrix}1&0\\0&1\end{bmatrix}$ has evals $\lambda = 1, 1$.

2) Short answer questions (4 pts each):

a) Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$. Calculate the area of the parallelogram formed by **u** and **v**.

$$A = \begin{bmatrix} \vec{u} & \vec{v} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ -1 & 2 \\ 3 & 0 \end{bmatrix} \qquad A^T A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ -1 & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 15 & 0 \\ 0 & 5 \end{bmatrix}$$

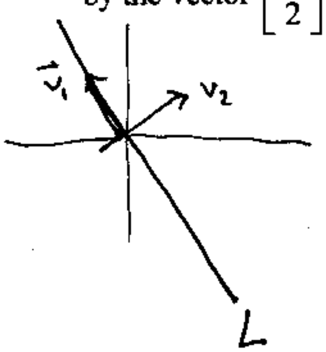
$$\text{Area} = \sqrt{\det(A^T A)} = \sqrt{75} = \boxed{5\sqrt{3}}$$

Also, $\vec{u} \cdot \vec{v} = 0 \Rightarrow \vec{u} \perp \vec{v}$, so area $= \|\vec{u}\| \|\vec{v}\|$

$$= \sqrt{1+4+1+9}\,\sqrt{0+1+4+0}$$

$$= \sqrt{15}\,\sqrt{5} = 5\sqrt{3}$$

b) Find the matrix representing the linear transformation from $\mathbf{R}^2$ to $\mathbf{R}^2$ that is reflection in the line spanned by the vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.



MANY GOOD APPROACHES. For example, let $\vec{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$

$$[A]_{\mathcal{B}} = S^{-1} A S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ where } S = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}.$$

$$\text{so } A = S \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} S^{-1} = \frac{1}{5} \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} = \boxed{\frac{1}{5} \begin{bmatrix} -3 & -4 \\ -4 & 3 \end{bmatrix} = A}$$

Problem 2 continues on next page

2) c) Find a matrix with eigenvalues equal to 2,3,5,7.

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

d) Consider the vector space V consisting of all 2×2 matrices for which the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector.

Find a basis for this space, and determine its dimension.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \begin{matrix} a + 2b = \lambda \\ c + 2d = 2\lambda \end{matrix} \Rightarrow \begin{matrix} a = \lambda - 2b \\ c = 2\lambda - 2d \end{matrix}$$

$$\Rightarrow \begin{bmatrix} \lambda - 2b & b \\ 2\lambda - 2d & d \end{bmatrix} = \lambda \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} + b \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}$$

<u>basis</u> dimension $= 3$.

Alternatively, $c + 2d = 2\lambda = 2a + 4b \Rightarrow c = 2a + 4b - 2d$

So matrix is of form

$$\begin{bmatrix} a & b \\ 2a+4b-2d & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}$$

basis.

3) (8 pts) Consider a linear transformation $T:\mathbf{R}^2 \to \mathbf{R}^2$. Suppose the matrix of T with respect to the basis $\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}1\\3\end{bmatrix}\right\}$ is $\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}$. Find the matrix of T with respect to the basis $\left\{\begin{bmatrix}1\\3\end{bmatrix},\begin{bmatrix}1\\4\end{bmatrix}\right\}$.

$\mathcal{B}_1 = \left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}1\\3\end{bmatrix}\right\}$, $\quad \left\{\begin{bmatrix}1\\3\end{bmatrix},\begin{bmatrix}1\\4\end{bmatrix}\right\} = \mathcal{B}_2$

$$S = \begin{bmatrix}1 & 1\\2 & 3\end{bmatrix} \qquad S^{-1}AS = \begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}$$

$$S^{-1} = \begin{bmatrix}3 & -1\\-2 & 1\end{bmatrix} \qquad A = S\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}S^{-1} = \begin{bmatrix}1 & 1\\2 & 3\end{bmatrix}\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}3 & -1\\-2 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1 & 1\\2 & 3\end{bmatrix}\begin{bmatrix}5 & -2\\1 & 0\end{bmatrix} = \begin{bmatrix}6 & -2\\13 & -4\end{bmatrix}$$

$$Q = \begin{bmatrix}1 & 1\\3 & 4\end{bmatrix} \qquad [A]_{\mathcal{B}_2} = Q^{-1}AQ = \begin{bmatrix}4 & -1\\-3 & 1\end{bmatrix}\begin{bmatrix}6 & -2\\13 & -4\end{bmatrix}\begin{bmatrix}1 & 1\\3 & 4\end{bmatrix}$$

$$Q^{-1} = \begin{bmatrix}4 & -1\\-3 & 1\end{bmatrix} \qquad = \begin{bmatrix}4 & -1\\-3 & 1\end{bmatrix}\begin{bmatrix}0 & -2\\1 & -3\end{bmatrix} = \boxed{\begin{bmatrix}-1 & -5\\1 & 3\end{bmatrix}}$$

4) (12 pts) Let **A** be a real $n \times n$ matrix such that $\mathbf{A}^2 = -\mathbf{I}_n$.

a) Show that **A** is invertible.

If $A^2 = -I$, then $A(-A) = -A^2 = I$

so $A^{-1} = -A$

and A is invertible.

b) Show that n must be even.

Since $A^{-1} = -A$, then

$$\det(A^{-1}) = \det(-A)$$

$$\frac{1}{\det A} = (-1)^n \det A$$

so $\frac{1}{\det A} = (-1)^n \det A \Rightarrow (\det A)^2 = (-1)^n$

LHS > 0. Therefore RHS > 0

$\Rightarrow$ n even.

c) Show that **A** has no real eigenvalues.

Suppose $A\vec{v} = \lambda\vec{v}$ for some eigenvector $\vec{v}$ with real eigenvalue λ.

Then $A^2\vec{v} = A(A\vec{v}) = A(\lambda\vec{v}) = \lambda A\vec{v} = \lambda^2\vec{v}$

But $A^2\vec{v} = -I\vec{v} = -\vec{v}$ $\Rightarrow$ $\lambda^2\vec{v} = -\vec{v}$

$\Rightarrow (\lambda^2 + 1)\vec{v} = \vec{0}$

Therefore $\lambda^2 + 1 = 0$

$\Rightarrow \lambda = \pm i$.

So λ is not real.

5) (12 pts) Consider the quadratic form $q(x_1,x_2,x_3)=2x_1^2+2x_2^2+2x_3^2-2x_1x_2-2x_2x_3+2x_1x_3$.

a) Find a symmetric matrix **A** such that $q(\mathbf{x})=\mathbf{x}^T\mathbf{A}\mathbf{x}$ for all $\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ in $\mathbf{R}^3$.

1 $$A=\begin{bmatrix}2&-1&1\\-1&2&-1\\1&-1&2\end{bmatrix}$$

b) Find all the eigenvalues of **A** and their algebraic and geometric multiplicities.

$$\lambda I-A=\begin{bmatrix}\lambda-2&1&-1\\1&\lambda-2&1\\-1&1&\lambda-2\end{bmatrix}$$

$$\begin{aligned}P_A(\lambda)&=(\lambda-2)(\lambda^2-4\lambda+3)-1(\lambda-2+1)-1(1+\lambda-2)\\&=\lambda^3-6\lambda^2+11\lambda-6-\lambda+1-\lambda+1\\&=\lambda^3-6\lambda^2+9\lambda-4=0\\&=(\lambda-1)(\lambda^2-5\lambda+4)=0\\&=(\lambda-1)(\lambda-1)(\lambda-4)\\&(\lambda-1)^2(\lambda-4)=0\end{aligned}$$

5 $\boxed{\text{evals are } \lambda_1=\lambda_2=1 \text{ Alg mult = Geom mult = 2 and } \lambda_3=4 \text{ Alg mult = Geom mult = 1}}$

c) Is **A** positive definite? Briefly justify your answer.

YES, all eigenvalues are strictly positive.

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d) Find an orthonormal eigenbasis for **A**.

$$\lambda_1=1\Rightarrow\left[\begin{array}{ccc|c}-1&1&-1&0\\1&-1&1&0\\-1&1&-1&0\end{array}\right]\rightarrow\left[\begin{array}{ccc|c}1&-1&1&0\\0&0&0&0\\0&0&0&0\end{array}\right]\qquad \begin{array}{l}x_1=s-t\\x_2=s\\x_3=t\end{array}\Rightarrow \vec{x}=s\overset{\vec{v}_1}{\begin{bmatrix}1\\1\\0\end{bmatrix}}+t\overset{\vec{v}_2}{\begin{bmatrix}-1\\0\\1\end{bmatrix}}$$

s t

not orthogonal. use Gram-Schmidt.

4 $$\vec{w}_1=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}\qquad \vec{v}_2-(\vec{v}_2\cdot\vec{w}_1)\vec{w}_1=\begin{bmatrix}-1\\0\\1\end{bmatrix}-\left(-\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$=\begin{bmatrix}-1\\0\\1\end{bmatrix}+\begin{bmatrix}1/2\\1/2\\0\end{bmatrix}=\begin{bmatrix}-1/2\\1/2\\1\end{bmatrix}=\frac{1}{2}\begin{bmatrix}-1\\1\\2\end{bmatrix}$$

normalize $\Rightarrow \vec{w}_2=\frac{1}{\sqrt{6}}\begin{bmatrix}-1\\1\\2\end{bmatrix}$.

$\lambda_3=4$

$$\Rightarrow\left[\begin{array}{ccc|c}2&1&-1&0\\1&2&1&0\\-1&1&2&0\end{array}\right]\rightarrow\left[\begin{array}{ccc|c}1&2&1&0\\0&3&3&0\\0&3&3&0\end{array}\right]\rightarrow\left[\begin{array}{ccc|c}1&0&-1&0\\0&1&1&0\\0&0&0&0\end{array}\right]\Rightarrow \vec{w}_3=\frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix}$$

ON BASIS IS

$$\left\{\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix},\begin{bmatrix}-1/\sqrt{6}\\1/\sqrt{6}\\2/\sqrt{6}\end{bmatrix},\begin{bmatrix}1/\sqrt{3}\\-1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}\right\}.$$

6) (12 pts) A rabbit population and a wolf population are modeled by the equations

$$r(t+1) = 5r(t) - 2w(t)$$
$$w(t+1) = r(t) + 2w(t)$$

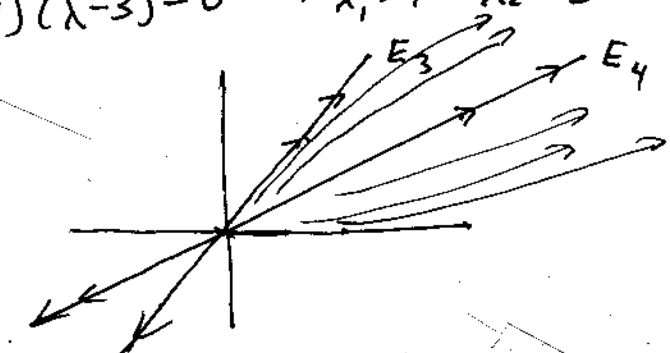
The initial populations are $r(0)$=300 and $w(0)$=200.

a) Find formulas for $r(t)$ and $w(t)$. Let $\vec{x} = \begin{bmatrix} r \\ w \end{bmatrix}$ $\quad \vec{x}(t+1) = A\,\vec{x}(t)$

where $A = \begin{bmatrix} 5 & -2 \\ 1 & 2 \end{bmatrix}$

$$\lambda I - A = \begin{bmatrix} \lambda-5 & 2 \\ -1 & \lambda-2 \end{bmatrix} \qquad P_A(\lambda) = \lambda^2 - 7\lambda + 12 = 0$$

$$(\lambda-4)(\lambda-3) = 0 \Rightarrow \lambda_1 = 4 \quad \lambda_2 = 3$$



$$\lambda_1 = 4 \Rightarrow \left[\begin{array}{cc|c} -1 & 2 & 0 \\ -1 & 2 & 0 \end{array}\right] \Rightarrow \vec{v}_1 = \begin{bmatrix} 2 \\ +1 \end{bmatrix}$$

$$\lambda_2 = 3 \Rightarrow \left[\begin{array}{cc|c} -2 & 2 & 0 \\ -1 & 1 & 0 \end{array}\right] \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$S = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \qquad S^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$S^{-1}AS = D = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} \qquad A = SDS^{-1} \qquad A^t = SD^tS^{-1} \qquad D^t = \begin{bmatrix} 4^t & 0 \\ 0 & 3^t \end{bmatrix}$$

$$\vec{x}(t) = A^t\,\vec{x}(0) = SD^tS^{-1}\vec{x}(0) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4^t & 0 \\ 0 & 3^t \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 300 \\ 200 \end{bmatrix}$$

$$= \begin{bmatrix} 2\cdot 4^t & 3^t \\ 4^t & 3^t \end{bmatrix}\begin{bmatrix} 100 \\ 100 \end{bmatrix} = \begin{bmatrix} 200\cdot 4^t + 100\cdot 3^t \\ 100\cdot 4^t + 100\cdot 3^t \end{bmatrix} = \begin{bmatrix} r(t) \\ w(t) \end{bmatrix}$$

b) In the long run, what will be the proportion of rabbits to wolves? Explain.

$$\frac{r(t)}{w(t)} = \frac{200\cdot 4^t + 100\cdot 3^t}{100\cdot 4^t + 100\cdot 3^t} = \frac{200 + 100\left(\frac{3}{4}\right)^t}{100 + 100\left(\frac{3}{4}\right)^t} \longrightarrow \frac{200}{100} = \frac{2}{1}$$

2 : 1 proportion of RABBITS TO WOLVES.

This is also clear from the eigenvector $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ corr. to the largest eigenvalue.

7) (12 pts) Let $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -b & -c \end{bmatrix}$ where b and c are real numbers.

Consider the continuous dynamical system $\dfrac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$.

a) What inequality or inequalities involving b and c ensure that the solutions to the system will consist of trajectories spiraling inwards toward the origin?

$$\lambda I - A = \begin{bmatrix} \lambda & -1 \\ b & \lambda + c \end{bmatrix} \qquad P_A(\lambda) = \lambda^2 + c\lambda + b = 0 \Rightarrow \lambda = \frac{-c \pm \sqrt{c^2 - 4b}}{2}$$

For spiral, e-vals must be complex, so $c^2 - 4b < 0 \Rightarrow \boxed{c^2 < 4b}$

3

Also, $Re(\lambda) < 0$ to spiral in, so $\boxed{c > 0}$

b) Solve this continuous dynamical system in the case where $b = 4$, $c = 5$, and $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(Your answer should be a closed formula for $\mathbf{x}(t)$.)

$$A = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \qquad \lambda I - A = \begin{bmatrix} \lambda & -1 \\ 4 & \lambda + 5 \end{bmatrix} \qquad P_A(\lambda) = \lambda^2 + 5\lambda + 4 = (\lambda + 4)(\lambda + 1) = 0$$

$$\lambda_1 = -4 \quad \lambda_2 = -1.$$

$$\lambda_1 = -4 \Rightarrow \left[\begin{array}{cc|c} -4 & -1 & 0 \\ 4 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1/4 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \vec{v}_1 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$$\lambda_2 = -1 \Rightarrow \left[\begin{array}{cc|c} -1 & -1 & 0 \\ 4 & 4 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \vec{v}_2 = \begin{bmatrix} +1 \\ -1 \end{bmatrix}$$

$$S = \begin{bmatrix} -1 & +1 \\ 4 & -1 \end{bmatrix} \quad S^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$$

$$S^{-1}AS = D = \begin{bmatrix} -4 & 0 \\ 0 & -1 \end{bmatrix}$$

6

$$A = SDS^{-1} \qquad [e^{tA}] = S[e^{tD}]S^{-1}$$

$$\vec{x}(t) = [e^{tA}]\vec{x}(0) = S\, e^{tD} S^{-1} \vec{x}(0)$$

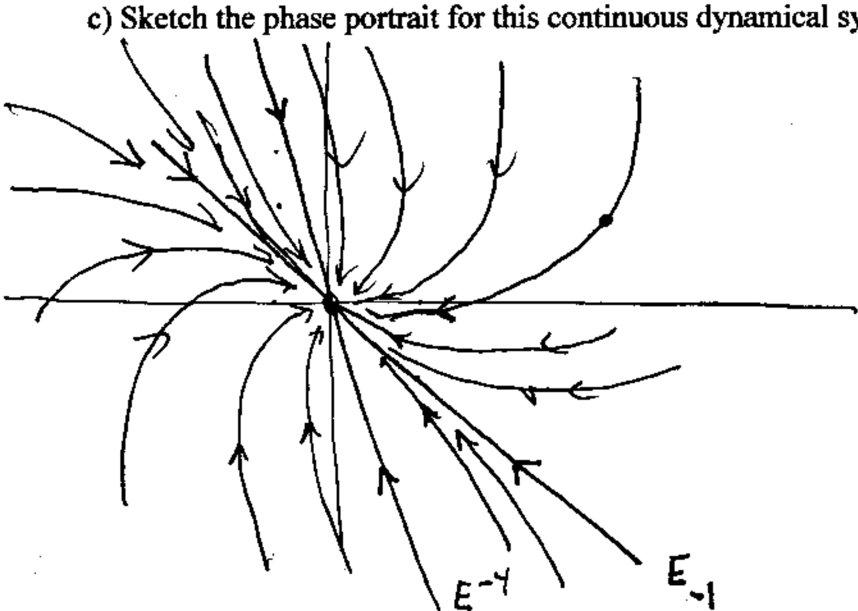
$$\frac{1}{3}\begin{bmatrix} -1 & 1 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} e^{-4t} & 0 \\ 0 & e^{-t} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} -e^{-4t} & e^{-t} \\ 4e^{-4t} & -e^{-t} \end{bmatrix}\begin{bmatrix} 3 \\ 9 \end{bmatrix} = \begin{bmatrix} -e^{-4t} & e^{-t} \\ 4e^{-4t} & -e^{-t} \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

c) Sketch the phase portrait for this continuous dynamical system.

$$\vec{x}(t) = \begin{bmatrix} -e^{-4t} + 3e^{-t} \\ 4e^{-4t} - 3e^{-t} \end{bmatrix}$$

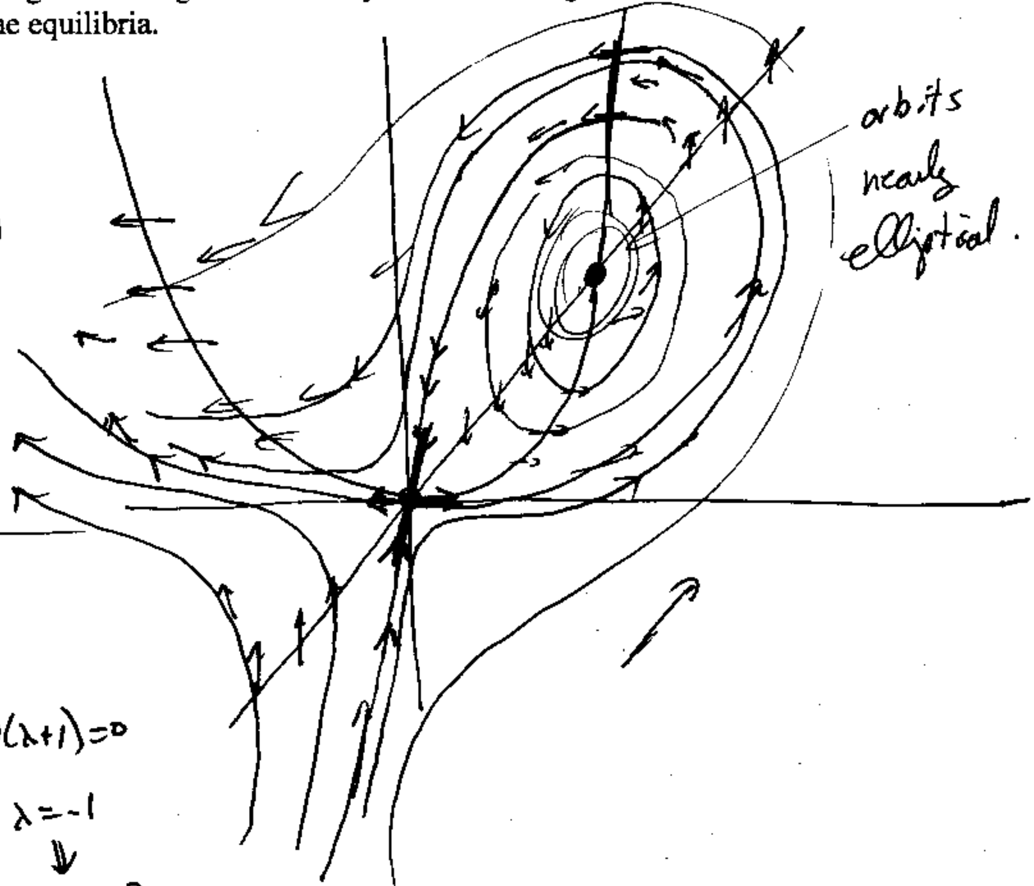
3



8) (12 pts) Consider the system $\left\{ \begin{array}{l} \dfrac{dx}{dt} = x - y \\ \dfrac{dy}{dt} = x^2 - y \end{array} \right\}$.

a) Perform the qualitative phase plane analysis for this system (i.e., find the null clines, equilibrium points, and general directions). Carry this out in the whole xy-plane (not just the first quadrant).

b) List the equilibrium points of the system above, and determine their stability. That is, linearize the system at each equilibrium and do the eigenvalue-eigenvector analysis. Give a rough sketch of some solutions, particularly in the vicinity of the equilibria.



Null clines

$\frac{dx}{dt} = 0 \Rightarrow \boxed{y = x \quad \text{VERTICAL}}$

$\frac{dy}{dt} = 0 \Rightarrow \boxed{y = x^2 \quad \text{HORIZONTAL}}$

EQUILIBRIA AT $(0,0)$, $(1,1)$

$$J_F = \begin{bmatrix} 1 & -1 \\ 2x & -1 \end{bmatrix}$$

$$J_F(0,0) = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$$

evals: $\begin{vmatrix} \lambda - 1 & 1 \\ 0 & \lambda + 1 \end{vmatrix} = (\lambda - 1)(\lambda + 1) = 0$

$\lambda = +1 \qquad \lambda = -1$

$\Downarrow \qquad \Downarrow$

$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 2 & 0 \end{array}\right] \qquad \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\Downarrow \qquad \Downarrow$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

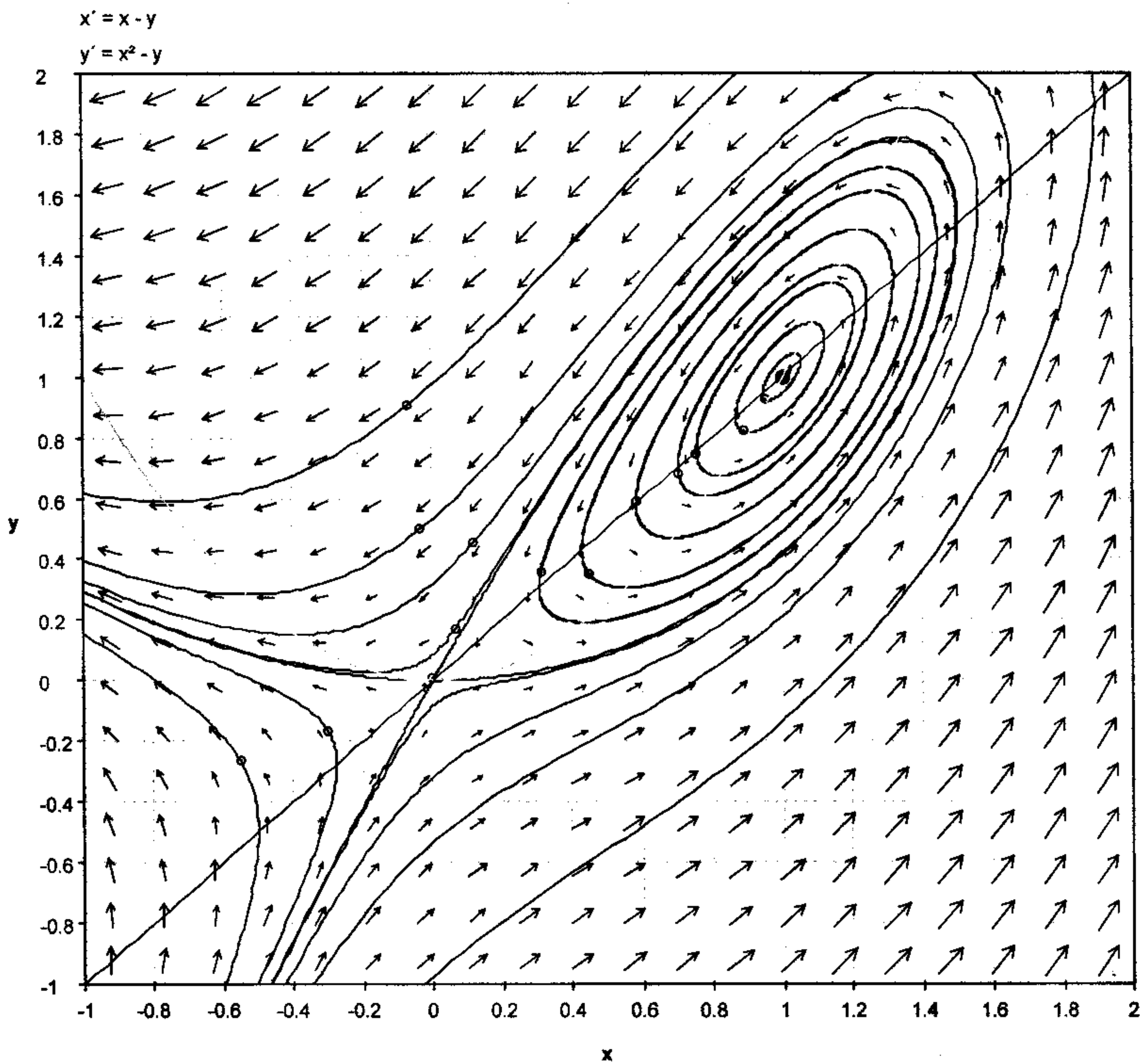
$$J_F(1,1) = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$$

evals: $\begin{vmatrix} \lambda - 1 & 1 \\ -2 & \lambda + 1 \end{vmatrix} \Rightarrow \lambda^2 + 1 = 0$

$\lambda = \pm i$

Linearized system will have elliptical (periodic) solutions.
However, we do not know if this is the case for the nonlinear system.

ORBITS near $(1,1)$ might be periodic or they could be slowly spiralling in or slowly spiralling out.


x' = x - y
y' = x² - y
y
x
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

BONUS QUESTION (6 pts)

Consider the two subspaces V_1 and V_2 of $\mathbf{R}^4$, where $V_1 = \text{span}\left\{\begin{bmatrix}1\\2\\3\\0\end{bmatrix},\begin{bmatrix}3\\4\\1\\2\end{bmatrix},\begin{bmatrix}1\\0\\2\\3\end{bmatrix}\right\}$ and $V_2 = \text{span}\left\{\begin{bmatrix}1\\2\\0\\1\end{bmatrix},\begin{bmatrix}2\\3\\1\\0\end{bmatrix}\right\}$.

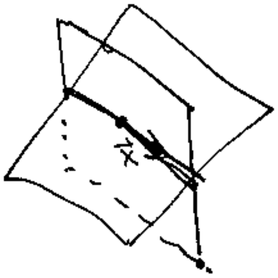
Find a basis for the intersection $V_1 \cap V_2$. [Note: The intersection of two subspaces is also a subspace.]

$$A=\begin{bmatrix}1&3&1\\2&4&0\\3&1&2\\0&2&3\end{bmatrix} \to \begin{bmatrix}1&3&1\\0&2&2\\0&8&1\\0&2&3\end{bmatrix} \to \begin{bmatrix}1&3&1\\0&1&1\\0&8&1\\0&2&3\end{bmatrix} \to \begin{bmatrix}1&0&-2\\0&1&1\\0&0&1\\0&0&0\end{bmatrix} \to \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix} \Rightarrow \text{lin. indep.}\quad \boxed{V_1 \text{ is 3-dim'l.}}$$

$$B=\begin{bmatrix}1&2\\2&3\\0&1\\1&0\end{bmatrix} \longrightarrow \boxed{V_2 \text{ clearly 2-dim'l}}$$

Let's characterize $V_1 \cap V_2$: If $\vec{x} \in V_1 \cap V_2$, then $\vec{x} \in V_1$ and $\vec{x} \in V_2$.

Look at orthogonal complements.



$$V_1^\perp = \ker(A^T) \quad \left[\begin{array}{cccc|c}1&2&3&0&0\\3&4&1&2&0\\1&0&2&3&0\end{array}\right] \to$$

DETAILS OMITTED

$$\to \left[\begin{array}{cccc|c}1&0&0&19/7&0\\0&1&0&-11/7&0\\0&0&1&-1/7&0\end{array}\right]$$

$$\vec{w}_3 = \begin{bmatrix}-19\\11\\-1\\7\end{bmatrix} \quad V_1^\perp = \text{Span}\{\vec{w}_1\}.$$

$$V_2^\perp = \ker(B^T) \Rightarrow \left[\begin{array}{cccc|c}1&2&0&1&0\\2&3&1&0&0\end{array}\right] \to \left[\begin{array}{cccc|c}1&2&0&1&0\\0&1&-1&2&0\end{array}\right] \to \left[\begin{array}{cccc|c}1&0&2&-3&0\\0&1&-1&2&0\end{array}\right]$$

$$\begin{array}{cc}\uparrow&\uparrow\\ s&t\end{array}$$

$x_1 = -2s+3t$
$x_2 = s-2t$
$x_3 = s$
$x_4 = t$

$$\Rightarrow \vec{x} = s\underbrace{\begin{bmatrix}-2\\1\\1\\0\end{bmatrix}}_{\vec{w}_1} + t\underbrace{\begin{bmatrix}3\\-2\\0\\1\end{bmatrix}}_{\vec{w}_2}. \qquad V_2^\perp = \text{Span}\{\vec{w}_2, \vec{w}_3\}.$$

$$(V_1 \cap V_2)^\perp = \text{Span}\{\vec{w}_1, \vec{w}_2, \vec{w}_3\} = \text{im}\, C \quad \text{where } C = \begin{bmatrix}-2&3&-19\\1&-2&11\\1&0&-1\\0&1&7\end{bmatrix}$$

$$\text{So } (V_1 \cap V_2) = (\text{im}\, C)^\perp = \ker(C^T)$$

$$\left[\begin{array}{cccc|c}-2&1&1&0&0\\3&-2&0&1&0\\-19&11&-1&7&0\end{array}\right] \to \cdots \to \left[\begin{array}{cccc|c}1&0&0&-13/3&0\\0&1&0&-7&0\\0&0&1&-5/3&0\end{array}\right] \Rightarrow \vec{K} = \begin{bmatrix}13\\21\\5\\3\end{bmatrix}$$

BASIS FOR $V_1 \cap V_2$