2.1

1. Not a linear transformation, since $y_2 = x_2 + 2$ is not linear in our sense.

2. Linear, with matrix $\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 1 & 0 & 0 \end{bmatrix}$

3. Not linear, since $y_2 = x_1 x_3$ is nonlinear

4. $A = \begin{bmatrix} 9 & 3 & -3 \\ 2 & -9 & 1 \\ 4 & -9 & -2 \\ 5 & 1 & 5 \end{bmatrix}$

5. By Fact 2.1.2, the three columns of the 2×3 matrix A are $T(\vec{e}_1), T(\vec{e}_2)$, and $T(\vec{e}_3)$, so that
$$A = \begin{bmatrix} 7 & 6 & -13 \\ 11 & 9 & 17 \end{bmatrix}.$$

6. Note that $x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, so that T is indeed linear, with matrix $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

7. Note that $x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = [\vec{v}_1 \ldots \vec{v}_n] \begin{bmatrix} x_1 \\ \cdots \\ x_n \end{bmatrix}$, so that T is indeed linear, with matrix $[\vec{v}_1 \; \vec{v}_2 \; \cdots \; \vec{v}_n]$.

8. Reducing the system $\left| \begin{array}{r} x_1 + 7x_2 = y_1 \\ 3x_1 + 20x_2 = y_2 \end{array} \right|$, we obtain $\left| \begin{array}{ccccc} x_1 & = & -20y_1 & + & 7y_2 \\ x_2 & = & 3y_1 & - & y_2 \end{array} \right|$.

9. We have to attempt to solve the equation $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for x_1 and x_2. Reducing the system
$\left| \begin{array}{ccccc} 2x_1 & + & 3x_2 & = & y_1 \\ 6x_1 & + & 9x_2 & = & y_2 \end{array} \right|$ we obtain $\left| \begin{array}{r} x_1 + 1.5x_2 = 0.5y_1 \\ 0 = -3y_1 + y_2 \end{array} \right|$.
No unique solution (x_1, x_2) can be found for a given (y_1, y_2); the matrix is noninvertible.

11. We have to attempt to solve the equation $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for x_1 and x_2. Reducing the system $\left| \begin{array}{ccccc} x_1 & + & 2x_2 & = & y_1 \\ 3x_1 & + & 9x_2 & = & y_2 \end{array} \right|$ we find that $\left| \begin{array}{ccc} x_1 & = & 3y_1 - \frac{2}{3}y_2 \\ x_2 & = & -y_1 + \frac{1}{3}y_2 \end{array} \right|$. The inverse matrix is $\begin{bmatrix} 3 & -\frac{2}{3} \\ -1 & \frac{1}{3} \end{bmatrix}$.

12. Reducing the system $\left| \begin{array}{c} x_1 + kx_2 = y_1 \\ x_2 = y_2 \end{array} \right|$ we find that $\left| \begin{array}{ccc} x_1 & = & y_1 - ky_2 \\ x_2 & = & y_2 \end{array} \right|$. The inverse matrix is $\begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}$.

13. a. First suppose that $a \neq 0$. We have to attempt to solve the equation $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for x_1 and x_2.

$$\left| \begin{array}{ccccc} ax_1 & + & bx_2 & = & y_1 \\ cx_1 & + & dx_2 & = & y_2 \end{array} \right| \div a \longrightarrow \left| \begin{array}{ccccc} x_1 & + & \frac{b}{a}x_2 & = & \frac{1}{a}y_1 \\ cx_1 & + & dx_2 & = & y_2 \end{array} \right| -c(I)$$

$$\left| \begin{array}{ccccc} x_1 & + & \frac{b}{a}x_2 & = & \frac{1}{a}y_1 \\ & & (d - \frac{bc}{a})x_2 & = & -\frac{c}{a}y_1 + y_2 \end{array} \right|$$

$$\left| \begin{array}{ccccc} x_1 & + & \frac{b}{a}x_2 & = & \frac{1}{a}y_1 \\ & & (\frac{ad-bc}{a})x_2 & = & -\frac{c}{a}y_1 + y_2 \end{array} \right|$$

We can solve this system for x_1 and x_2 if (and only if) $ad - bc \neq 0$, as claimed.
If $a = 0$, then we have to consider the system

$$\left| \begin{array}{ccccc} & & bx_2 & = & y_1 \\ cx_1 & + & dx_2 & = & y_2 \end{array} \right| \xrightarrow{I \leftrightarrow II} \left| \begin{array}{ccccc} cx_1 & + & dx_2 & = & y_2 \\ & & bx_2 & = & y_1 \end{array} \right|$$

We can solve for x_1 and x_2 provided that both b and c are nonzero, that is if $bc \neq 0$. Since $a = 0$, this means that $ad - bc \neq 0$, as claimed.

b. First suppose that $ad - bc \neq 0$ and $a \neq 0$. Let $D = ad - bc$ for simplicity. We continue our work in part (a):

$$\left| \begin{array}{ccccc} x_1 & + & \frac{b}{a}x_2 & = & \frac{1}{a}y_1 \\ & & \frac{D}{a}x_2 & = & -\frac{c}{a}y_1 + y_2 \end{array} \right| \cdot \frac{a}{D}$$

$$\left| \begin{array}{ccccc} x_1 & + & \frac{b}{a}x_2 & = & \frac{1}{a}y_1 \\ & & x_2 & = & -\frac{c}{D}y_1 + \frac{a}{D}y_2 \end{array} \right| -\frac{b}{a}(II)$$

$$\left| \begin{array}{ccc} x_1 & = & (\frac{1}{a} + \frac{bc}{aD})y_1 - \frac{b}{D}y_2 \\ x_2 & = & -\frac{c}{D}y_1 + \frac{a}{D}y_2 \end{array} \right|$$

$$\left| \begin{array}{ccc} x_1 & = & \frac{d}{D}y_1 - \frac{b}{D}y_2 \\ x_2 & = & -\frac{c}{D}y_1 + \frac{a}{D}y_2 \end{array} \right|$$

$$\left(\text{Note that } \frac{1}{a} + \frac{bc}{aD} = \frac{D + bc}{aD} = \frac{ad}{aD} = \frac{d}{D}. \right)$$

It follows that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, as claimed. If $ad - bc \neq 0$ and $a = 0$, then we have to solve the system

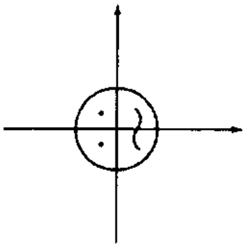
$$\left|\begin{array}{rl} cx_1 + dx_2 = y_2 \\ bx_2 = y_1 \end{array}\right| \begin{array}{l} \div c \\ \div b \end{array}$$

$$\left|\begin{array}{rl} x_1 + \frac{d}{c}x_2 = \frac{1}{c}y_2 \\ x_2 = \frac{1}{b}y_1 \end{array}\right| \quad -\frac{d}{c}(II)$$

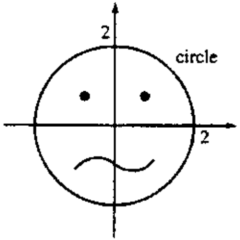
$$\left|\begin{array}{cccc} x_1 & & = & -\frac{d}{bc}y_1 \quad +\frac{1}{c}y_2 \\ & x_2 & = & \frac{1}{b}y_1 \end{array}\right|$$

It follows that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{d}{bc} & \frac{1}{c} \\ \frac{1}{b} & 0 \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ (recall that $a = 0$), as claimed.

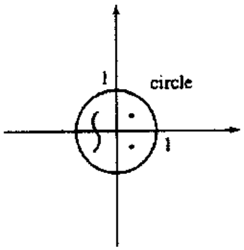
24. Compare with Example 5.



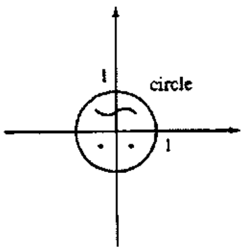
25. The matrix represents a dilation by the factor of 2.



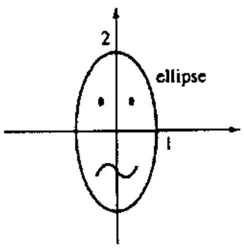
26. Matrix represents a reflection in the line $x_2 = x_1$.



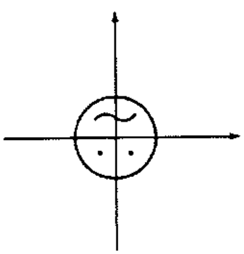
27. Matrix represents a reflection in the $\vec{e}_1$ axis.



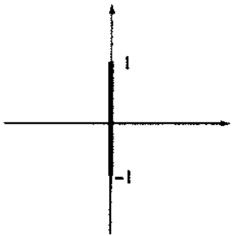
28. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, then $A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_2 \end{bmatrix}$, so that the x_2 component is multiplied by 2, while the x_1 component remains unchanged.



29. Matrix represents a reflection in the origin. Compare with Exercise 17.



30. If $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, then $A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$, so that A represents the projection onto the $\vec{e}_2$ axis.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-1

43. a. $T(\vec{x}) = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2x_1 + 3x_2 + 4x_3 = [2\ 3\ 4] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

The transformation is indeed linear, with matrix $[2\ 3\ 4]$.

b. If $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, then T is linear with matrix $[v_1\ v_2\ v_3]$, as in part (a).

c. Let $[a\ b\ c]$ be the matrix of T. Then $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [a\ b\ c] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = ax_1 + bx_2 + cx_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, so that $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ does the job.

44. $T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} v_2x_3 - v_3x_2 \\ v_3x_1 - v_1x_3 \\ v_1x_2 - v_2x_1 \end{bmatrix} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, so that T is linear,

with matrix $\begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$.