5. Find all $\vec{x}$ such that $A\vec{x} = \vec{0}$.

$$\begin{bmatrix} 1 & 1 & 1 & \vdots & 0 \\ 1 & 2 & 3 & \vdots & 0 \\ 1 & 3 & 5 & \vdots & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & \vdots & 0 \\ 0 & 1 & 2 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}; \begin{matrix} x_1 = x_3 \\ x_2 = -2x_3 \end{matrix}; \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix}$$

$$\ker(A) = \operatorname{span} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

6. Find all $\vec{x}$ such that $A\vec{x} = \vec{0}$.

$$\begin{bmatrix} 1 & 1 & 1 & \vdots & 0 \\ 1 & 1 & 1 & \vdots & 0 \\ 1 & 1 & 1 & \vdots & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}; x_1 + x_2 + x_3 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -r-t \\ r \\ t \end{bmatrix} = r\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\ker(A) = \operatorname{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right).$$

7. Find all $\vec{x}$ such that $A\vec{x} = \vec{0}$. Since $\operatorname{rref}(A) = I_3$ we have $\ker(A) = \{\vec{0}\}$.

8. Find all $\vec{x}$ such that $A\vec{x} = \vec{0}$. Solving this system yields $\ker(A) = \operatorname{span}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

9. Find all $\vec{x}$ such that $A\vec{x} = \vec{0}$. Solving this system yields $\ker(A) = \{\vec{0}\}$.

10. Solving the system $A\vec{x} = \vec{0}$ we find that $\ker(A) = \operatorname{span}\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$.

11. Solving the system $A\vec{x} = \vec{0}$ we find that $\ker(A) = \operatorname{span}\begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$.

12. Solving the system $A\vec{x} = \vec{0}$ we find that $\ker(A) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}\right)$.

13. Solving the system $A\vec{x} = \vec{0}$ we find that $\ker(A) = \operatorname{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right)$.

14. By Fact 3.1.3, the image of A is the span of the columns of A:

$$\operatorname{im}(A) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}\right).$$

17. By Fact 3.1.3, $\text{im}(A) = \text{span}\left(\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}\right) = \mathbb{R}^2$ (the whole plane).

18. By Fact 3.1.3, $\text{im}(A) = \text{span}\left(\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}4\\12\end{bmatrix}\right) = \text{span}\begin{bmatrix}1\\3\end{bmatrix}$ (a line in $\mathbb{R}^2$).

19. Since the four column vectors of A are parallel, we have $\text{im}(A) = \text{span}\begin{bmatrix}1\\-2\end{bmatrix}$, a line in $\mathbb{R}^2$.

20. Since the three column vectors of A are parallel, we have $\text{im}(A) = \text{span}\begin{bmatrix}1\\1\\1\end{bmatrix}$, a line in $\mathbb{R}^3$.

30. By Fact 3.1.3, $A = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ does the job. There are many other possible answers: any nonzero $2 \times n$ matrix A whose column vectors are scalar multiples of vector $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

31. The plane $x + 3y + 2z = 0$ is spanned by the two vectors $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$, for example. Therefore,

$A = \begin{bmatrix} -2 & -3 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ does the job. There are many other correct answers.

32. By Fact 3.1.3, $A = \begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}$ does the job. There are many other correct answers: any nonzero $3 \times n$ matrix A whose column vectors are scalar multiples of $\begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}$.

33. The plane is the kernel of the linear transformation $T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x + 2y + 3z$ from $\mathbb{R}^3$ to $\mathbb{R}$.

34. To describe a subset of $\mathbb{R}^3$ as a kernel means to describe it as an intersection of planes (think about it). By inspection, the given line is the intersection of the planes

$x + y = 0$ and
$2x + z = 0$.

This means that the line is the kernel of the linear transformation $T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + y \\ 2x + z \end{bmatrix}$ from $\mathbb{R}^3$ to $\mathbb{R}^2$.

44. a. Yes; by construction of the echelon form, the systems $A\vec{x} = \vec{0}$ and $B\vec{x} = \vec{0}$ have the same solutions (it is the whole point of Gaussian elimination not to change the solutions of a system).

b. No; as a counterexample, consider $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, with $\text{im}(A) = \text{span}(\vec{e}_2)$, but $B = \text{rref}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, with $\text{im}(B) = \text{span}(\vec{e}_1)$.