3.3

1. $\text{rref}(A) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

   A basis of $\ker(A)$ is $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$; $\dim(\ker(A)) = 1$.

2. $\text{rref}(A) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

   A basis of $\ker(A)$ is $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. Dimension is 2.

6. $\text{rref}(A) = I_3$
The basis of $\ker(A)$ is $\emptyset$, so that $\dim(\ker(A)) = 0$.

7. $\text{rref}(A) = A$
A basis of $\ker(A)$ is $\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-3\\0\\-4\\1\end{bmatrix}$, so that $\dim(\ker(A)) = 3$.

8. $\text{rref}(A) = \begin{bmatrix}1 & 2 & 0 & 5 & -2\\0 & 0 & 1 & -1 & 1\\0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0\end{bmatrix}$
A basis of $\ker(A)$ is $\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-5\\0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}2\\0\\-1\\0\\1\end{bmatrix}$, so that $\dim(\ker(A)) = 3$.

9. $\text{rref}(A) = A$
A basis of $\ker(A)$ is $\begin{bmatrix}1\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\0\\0\\1\end{bmatrix}$, so that $\dim(\ker(A)) = 4$.

10. $\text{rref}(A) = \begin{bmatrix}1 & 0 & 2 & 0\\0 & 1 & -3 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{bmatrix}$
A basis of $\ker(A)$ is $\begin{bmatrix}-2\\3\\1\\0\end{bmatrix}$, so that $\dim(\ker(A)) = 1$.

11. A basis of $\text{im}(A)$ is $\begin{bmatrix}1\\2\end{bmatrix}$, so that $\dim(\text{im}(A)) = 1$ (see Exercise 1).

12. A basis of $\text{im}(A)$ is $\begin{bmatrix}1\\1\\1\end{bmatrix}$, so that $\dim(\text{im}(A)) = 1$ (see Exercise 2).

13. A basis of $\text{im}(A)$ is $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}3\\2\\1\end{bmatrix}$, so that $\dim(\text{im}(A)) = 2$ (see Exercise 4).

16. A basis of im(A) is $\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\begin{bmatrix}3\\2\\1\end{bmatrix}$, $\begin{bmatrix}2\\3\\3\end{bmatrix}$, so that $\dim(\text{im}(A)) = 3$ (see Exercise 6).

17. $\text{rref}(A) = \begin{bmatrix}1 & 2 & 0\\0 & 0 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$, so that a basis of im($A$) is $\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, $\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$, with $\dim(\text{im}(A)) = 2$.

18. $\text{rref}(A) = \begin{bmatrix}1 & 2 & 0 & 0 & 2\\0 & 0 & 1 & 0 & -3\\0 & 0 & 0 & 1 & 1\\0 & 0 & 0 & 0 & 0\end{bmatrix}$, so that a basis of im($A$) is $\begin{bmatrix}4\\3\\2\\1\end{bmatrix}$, $\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, $\begin{bmatrix}1\\2\\9\\1\end{bmatrix}$, with $\dim(\text{im}(A)) = 3$.

19. $\text{rref}(A) = \begin{bmatrix}1 & 0 & 2 & 0\\0 & 1 & -3 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{bmatrix}$, so that a basis of im($A$) is $\begin{bmatrix}1\\0\\3\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\\4\\-1\end{bmatrix}$, $\begin{bmatrix}4\\-1\\8\\4\end{bmatrix}$, with $\dim(\text{im}(A)) = 3$.

20. A basis of im(A) is $\begin{bmatrix}1\\3\\1\\2\end{bmatrix}$, $\begin{bmatrix}3\\9\\4\\9\end{bmatrix}$, so that $\dim(\text{im}(A)) = 2$ (see Exercise 8).

24. We form a 5×5 matrix A with the given vectors as its columns, and we find a basis of $\text{im}(A)$.

$$\text{rref}(A) = \begin{bmatrix} 1 & 3 & 0 & -1 & 3 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis of $\text{im}(A)$ is $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 4 \\ 1 \\ 2 \end{bmatrix}$.

26. Find a basis $\vec{v}_1, \vec{v}_2, \vec{v}_3$ of $\ker(A)$ and let $B = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$.

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \\ 0 & 0 & 1 \end{bmatrix}.$$

32. We need to find all vectors $\vec{x}$ in $\mathbb{R}^4$ such that $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = 0$ and $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = 0$.

This amounts to solving the system $\left| \begin{array}{ccccccc} x_1 & & - & x_3 & + & x_4 & = 0 \\ & x_2 & + & 2x_3 & + & 3x_4 & = 0 \end{array} \right|$, which in turn amounts to finding the kernel of $\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}$.

Using the standard approach, we find the basis $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \end{bmatrix}$.

33. We can write $V = \ker(A)$, where A is the $1 \times n$ matrix $A = [c_1\ c_2\ \cdots\ c_n]$.
Since at least one of the c_i is nonzero, the rank of A is 1, so that $\dim(V) = \dim(\ker(A)) = n - \text{rank}(A) = n - 1$, by Fact 3.3.5.
A "hyperplane" in $\mathbb{R}^2$ is a line, and a "hyperplane" in $\mathbb{R}^3$ is just a plane.

34. We can write $V = \ker(A)$, where A is the $m \times n$ matrix with entries a_{ij}. Note that $\text{rank}(A) \leq m$. Therefore, $\dim(V) = \dim(\ker(A)) = n - \text{rank}(A) \geq n - m$, by Fact 3.3.5.

35. We need to find all vectors $\vec{x}$ in $\mathbb{R}^n$ such that $\vec{v} \cdot \vec{x} = 0$, or $v_1x_1 + v_2x_2 + \cdots + v_nx_n = 0$, where the v_i are the components of the vector $\vec{v}$. These vectors form a hyperplane in $\mathbb{R}^n$ (see Exercise 33), so that the dimension of the space is $n - 1$.

36. No; if $\text{im}(A) = \ker(A)$ for an $n \times n$ matrix A, then $n = \dim(\ker(A)) + \dim(\text{im}(A)) = 2\dim(\text{im}(A))$, so that n is an even number.