3.4

1. We need to find the scalars c_1 and c_2 such that $\begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. Solving a linear system gives $c_1 = 3$, $c_2 = 4$. Thus $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

2. Proceeding as in Example 1, we find $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

3. Proceeding as in Example 1, we find $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$.

4. Proceeding as in Example 1, we find $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$.

12. Proceeding as in Exercise 11, we find the coordinate vector $\begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$.

13. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $S = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $S^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$.

By Fact 3.4.4 the new matrix of T, namely B, is given by $B = S^{-1}AS = \begin{bmatrix} -1 & -1 \\ 4 & 6 \end{bmatrix}$.

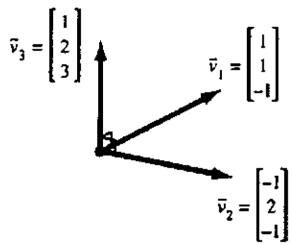
14. $A = \begin{bmatrix} 7 & -1 \\ -6 & 8 \end{bmatrix}$; $S = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$; $S^{-1} = \frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$.

By Fact 3.4.4 the new matrix of T, namely B, is given by $B = S^{-1}AS = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}$.

15. a. Let B be the matrix of T with respect to the basis $\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$. By Fact 3.4.3, $B = [[T(\vec{v}_1)]_B \quad [T(\vec{v}_2)]_B] = [[\vec{v}_1]_B \quad [\vec{0}]_B] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

b. $A = SBS^{-1}$, with the matrix $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ from part a, and $S = [\vec{v}_1 \ \vec{v}_2] = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$. A straightforward computation gives $A = \begin{bmatrix} 0.1 & 0.3 \\ 0.3 & 0.9 \end{bmatrix}$.

16. a. $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$ are on the plane, and $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is perpendicular to it.



Let B be the desired matrix. By Fact 3.4.3,

$$B = [[T(\vec{v}_1)]_B \ [T(\vec{v}_2)]_B \ [T(\vec{v}_3)]_B] = [[\vec{v}_1]_B [\vec{v}_2]_B [\vec{0}]_B] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

b. By Fact 3.4.4, $A = SBS^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 2 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 2 \\ -1 & -1 & 3 \end{bmatrix}^{-1} = \frac{1}{14} \begin{bmatrix} 13 & -2 & -3 \\ -2 & 10 & -6 \\ -3 & -6 & 5 \end{bmatrix}$.

17. By Fact 3.4.4, $B = S^{-1}AS = \begin{bmatrix} 3 & 5 \\ 5 & 8 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 9 \\ 9 & 4 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 5 & 8 \end{bmatrix} = \begin{bmatrix} -149 & -231 \\ 99 & 154 \end{bmatrix}$.

18. By Fact 3.4.4, $A = SBS^{-1} = \begin{bmatrix} 3 & 5 \\ 5 & 8 \end{bmatrix} \begin{bmatrix} 1 & 9 \\ 9 & 7 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 5 & 8 \end{bmatrix}^{-1} = \begin{bmatrix} -74 & 54 \\ -111 & 82 \end{bmatrix}$.

19. By Fact 3.4.4, $A = SBS^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} d & c \\ b & a \end{bmatrix}$.

20. By Fact 3.4.1, $\begin{bmatrix} 5 \\ 3 \end{bmatrix}_B = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

36. We seek a basis $\vec{v}_1 = \begin{bmatrix} x \\ z \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} y \\ t \end{bmatrix}$ such that the matrix $S = [\vec{v}_1 \ \vec{v}_2] = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ satisfies the equation $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & y \\ z & t \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$. Solving the ensuing linear system gives $S = \begin{bmatrix} \frac{z}{2} & -t \\ z & t \end{bmatrix}$. We need to choose both z and t nonzero to make S invertible. For example, if we let $z = 2$ and $t = 1$, then $S = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$, so that $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.