4.1

1. Not a subspace since it does not contain the neutral element, that is, the function $f(t) = 0$, for all t.

2. This subset V is a subspace of P_2:

   - The neutral element $f(t) = 0$ (for all t) is in V.

   - If f and g are in V (so that $f(2) = g(2) = 0$), then $(f+g)(2) = f(2) + g(2) = 0 + 0 = 0$, so that $f+g$ is in V.

   - If f is in V (so that $f(2) = 0$), and k is any constant, then $(kf)(2) = kf(2) = 0$, so that kf is in V.

   A polynomial $f(t) = a + bt + ct^2$ is in V if $f(2) = a + 2b + 4c = 0$, or $a = -2b - 4c$. The general element of V is of the form $f(t) = (-2b-4c) + bt + ct^2 = b(t-2) + c(t^2-4)$, so that $t-2$, t^2-4 is a basis of V.

3. This subset V is a subspace of P_2:

   - The neutral element $f(t) = 0$ (for all t) is in V since $f'(1) = f(2) = 0$.

   - If f and g are in V (so that $f'(1) = f(2)$ and $g'(1) = g(2)$), then $(f+g)'(1) = (f'+g')(1) = f'(1) + g'(1) = f(2) + g(2) = (f+g)(2)$, so that $f+g$ is in V.

   - If f is in V (so that $f'(1) = f(2)$) and k is any constant, then $(kf)'(1) = (kf')(1) = kf'(1) = kf(2) = (kf)(2)$, so that kf is in V.

   If $f(t) = a + bt + ct^2$ then $f'(t) = b + 2ct$, and f is in V if $f'(1) = b + 2c = a + 2b + 4c = f(2)$, or $a + b + 2c = 0$. The general element of V is of the form $f(t) = (-b-2c) + bt + ct^2 = b(t-1) + c(t^2-2)$, so that $t-1, t^2-2$ is a basis of V.

4. This subset V is a subspace of P_2:

   - The neutral element $f(t) = 0$ (for all t) is in V since $\int_0^1 0\,dt = 0$.

   - If f and g are in V $\left(\text{so that } \int_0^1 f = \int_0^1 g = 0\right)$ then $\int_0^1 (f+g) = \int_0^1 f + \int_0^1 g = 0$, so that $f+g$ is in V.

   - If f is in V $\left(\text{so that } \int_0^1 f = 0\right)$ and k is any constant, then $\int_0^1 kf = k\int_0^1 f = 0$, so that kf is in V.

   If $f(t) = a + bt + ct^2$ then $\int_0^1 f(t)dt = \left[at + \frac{b}{2}t^2 + \frac{c}{3}t^3\right]_0^1 = a + \frac{b}{2} + \frac{c}{3} = 0$ if $a = -\frac{b}{2} - \frac{c}{3}$.

The general element of V is $f(t) = \left(-\frac{b}{2} - \frac{c}{3}\right) + bt + ct^2 = b\left(t - \frac{1}{2}\right) + c\left(t^2 - \frac{1}{3}\right)$, so that $t - \frac{1}{2}, t^2 - \frac{1}{3}$ is a basis of V.

5. If $p(t) = a + bt + ct^2$ then $p(-t) = a - bt + ct^2$ and $-p(-t) = -a + bt - ct^2$.
Comparing coefficients we see that $p(t) = -p(-t)$ for all t if (and only if) $a = c = 0$.
The general element of the subset is of the form $p(t) = bt$.
These polynomials form a subspace of P_2, with basis t.

6. Not a subspace, since I_3 and $-I_3$ are invertible, but their sum is not.

7. The set V of diagonal 3×3 matrices is a subspace of $\mathbb{R}^{3\times3}$:

   a. The zero matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is in V,

   b. If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ and $B = \begin{bmatrix} p & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & r \end{bmatrix}$ are in V, then so is their sum

   $$A + B = \begin{bmatrix} a+p & 0 & 0 \\ 0 & b+q & 0 \\ 0 & 0 & c+r \end{bmatrix}.$$

   c. If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ is in V, then so is $kA = \begin{bmatrix} ka & 0 & 0 \\ 0 & kb & 0 \\ 0 & 0 & kc \end{bmatrix}$, for all constants k.

8. This is a subspace; the justification is analogous to Exercise 7.

9. Not a subspace; consider multiplication with a negative scalar. I_3 belongs to the set, but $-I_3$ doesn't.

10. Let $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Let V be the set of all 3×3 matrices A such that $A\vec{v} = \vec{0}$. Then V is a subspace of $\mathbb{R}^{3\times3}$:

    a. The zero matrix 0 is in V, since $0\vec{v} = \vec{0}$.

    b. If A and B are in V, then so is $A + B$, since $(A + B)\vec{v} = A\vec{v} + B\vec{v} = \vec{0} + \vec{0} = \vec{0}$.

    c. If A is in V, then so is kA for all scalars k, since $(kA)\vec{v} = k(A\vec{v}) = k\vec{0} = \vec{0}$.

18. Any f in P_n can be written as a linear combination of $1, t, t^2, \ldots, t^n$, by definition of P_n. Also, $1, t, \ldots, t^n$ are linearly independent; to see this consider a relation $c_0 + c_1 t + \cdots + c_n t^n = 0$; since the polynomial $c_0 + c_1 t + \cdots + c_n t^n$ has more than n zeros, we must have $c_0 = c_1 = \cdots = c_n = 0$, as claimed. Thus, $\dim(P_n) = n + 1$.

19. $\begin{bmatrix} a+bi \\ c+di \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} i \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 1 \end{bmatrix} + d\begin{bmatrix} 0 \\ i \end{bmatrix}$

The vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} i \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ i \end{bmatrix}$ form a basis of $\mathbb{C}^2$ as a *real* linear space, so that $\dim(\mathbb{C}^2) = 4$.

20. Use the strategy outlined on Page 155. We have $a = -d$, so that the general element of the subspace is $\begin{bmatrix} -d & b \\ c & d \end{bmatrix} = b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

Thus $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ is a basis of the subspace; the dimensions is 3.

28. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is in the subspace if $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix}$ equals $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}$

which is the case if $c = 0$ and $a = d$.

The matrices in the subspace are of the form $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, so that $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is a basis, and the dimension is 2.