12. $\|\vec{v}+\vec{w}\|^2 = (\vec{v}+\vec{w})\cdot(\vec{v}+\vec{w})$ (by hint)

$= \|\vec{v}\|^2 + \|\vec{w}\|^2 + 2(\vec{v}\cdot\vec{w})$ (by definition of length)

$\leq \|\vec{v}\|^2 + \|\vec{w}\|^2 + 2\|\vec{v}\|\|\vec{w}\|$ (by Cauchy-Schwarz)

$= (\|\vec{v}\| + \|\vec{w}\|)^2$, so that

$\|\vec{v}+\vec{w}\|^2 \leq (\|\vec{v}\| + \|\vec{w}\|)^2$

Taking square roots of both sides, we find that $\|\vec{v}+\vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$, as claimed.

15. The subspace consists of all vectors $\vec{x}$ in $\mathbb{R}^4$ such that $\vec{x}\cdot\vec{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = x_1 + 2x_2 + 3x_3 + 4x_4 = 0.$

These are vectors of the form $\begin{bmatrix} -2r & -3s & -4t \\ r & & \\ & s & \\ & & t \end{bmatrix} = r\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$

The three vectors to the right give the desired basis.

16. You may be able to find the solutions by educated guessing. Here is the systematic approach: we first find all vectors $\vec{x}$ that are orthogonal to $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$, then we identify the unit vectors among them. Finding the vectors $\vec{x}$ with $\vec{x}\cdot\vec{v}_1 = \vec{x}\cdot\vec{v}_2 = \vec{x}\cdot\vec{v}_3 = 0$ amounts to solving the system

$$\begin{vmatrix} x_1 - x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 - x_3 - x_4 = 0 \\ x_1 - x_2 + x_3 - x_4 = 0 \end{vmatrix}$$

(we can omit all the coefficients $\frac{1}{2}$).

The solutions are of the form $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ -t \\ t \end{bmatrix}.$

Since $\|\vec{x}\| = 2|t|$, we have a unit vector if $t = \frac{1}{2}$ or $t = -\frac{1}{2}$. Thus there are two possible choices for $\vec{v}_4$:

$$\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \text{ and } \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}.$$

17. The orthogonal complement $W^\perp$ of W consists of the vectors $\vec{x}$ in $\mathbb{R}^4$ such that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = 0 \text{ and } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} = 0.$$

Finding these vectors amounts to solving the system $\left| \begin{array}{l} x_1 + 2x_2 + 3x_3 + 4x_4 = 0 \\ 5x_1 + 6x_2 + 7x_3 + 8x_4 = 0 \end{array} \right|$.

The solutions are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} s + 2t \\ -2s - 3t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

The two vectors to the right form a basis of $W^\perp$.

26. The two given vectors spanning the subspace are orthogonal, but they are not unit vectors: both have length 7. To obtain an orthonormal basis $\vec{v}_1, \vec{v}_2$ of the subspace, we divide by 7:

$$\vec{v}_1 = \frac{1}{7}\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}, \vec{v}_2 = \frac{1}{7}\begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}.$$

Now we can use Fact 5.1.6, with $\vec{x} = \begin{bmatrix} 49 \\ 49 \\ 49 \end{bmatrix}$:

$$\text{proj}_V \vec{x} = (\vec{v}_1 \cdot \vec{x})\vec{v}_1 + (\vec{v}_2 \cdot \vec{x})\vec{v}_2 = 11\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} - \begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix} = \begin{bmatrix} 19 \\ 39 \\ 64 \end{bmatrix}.$$

27. Since the two given vectors in the subspace are orthogonal, we have the orthonormal basis

$$\vec{v}_1 = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \frac{1}{3}\begin{bmatrix} -2 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

Now we can use Fact 5.1.6, with $\vec{x} = 9\vec{e}_1$: $\text{proj}_V \vec{x} = (\vec{v}_1 \cdot \vec{x})\vec{v}_1 + (\vec{v}_2 \cdot \vec{x})\vec{v}_2 = 2\begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} -2 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ 2 \\ -2 \end{bmatrix}.$

28. Since the three given vectors in the subspace are orthogonal, we have the orthonormal basis

$$\vec{v}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \vec{v}_3 = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

Now we can use Fact 5.1.6, with $\vec{x} = \vec{e}_1$: $\text{proj}_V \vec{x} = (\vec{v}_1 \cdot \vec{x})\vec{v}_1 + (\vec{v}_2 \cdot \vec{x})\vec{v}_2 + (\vec{v}_3 \cdot \vec{x})\vec{v}_3 = \frac{1}{4}\begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}$.