5. $V = \ker(A)$, where $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 5 & 4 \end{bmatrix}$.

Then $V^\perp = (\ker A)^\perp = \operatorname{im}(A^T)$, by Exercise 4.

The two columns of A^T form a basis of $V^\perp$:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 5 \\ 4 \end{bmatrix}$$

6. Yes! For any matrix A,

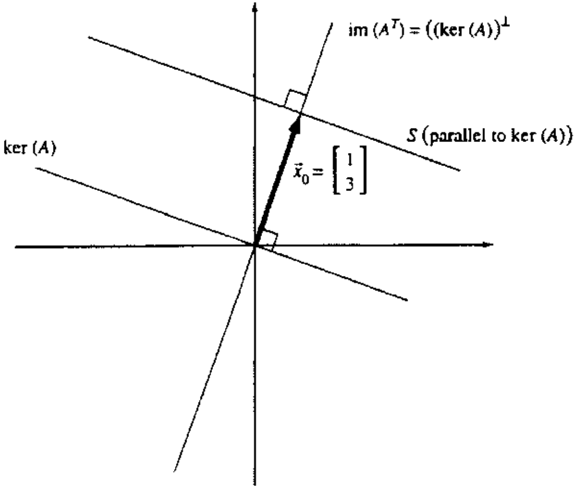
$$\operatorname{im}(A) = (\ker(A^T))^\perp = (\ker(AA^T))^\perp = (\ker(AA^T)^T)^\perp = \operatorname{im}(AA^T).$$

$\uparrow$ Facts 5.4.1, Facts 5.4.2b; $\uparrow$ Fact 5.4.3a; $\uparrow$ Facts 5.4.1 and 5.4.2b.

7. $\operatorname{im}(A)$ and $\ker(A)$ are orthogonal complements by Fact 5.4.1:

$(\operatorname{im} A)^\perp = \ker(A^T) = \ker(A)$

9.



$\vec{x}_0$ is the shortest of all the vectors in S.

10. a. If $\vec{x}$ is an arbitrary solution of the system $A\vec{x} = \vec{b}$, let $\vec{x}_h = \text{proj}_V \vec{x}$, where $V = \ker(A)$, and $\vec{x}_0 = \vec{x} - \text{proj}_V \vec{x}$. Note that $\vec{b} = A\vec{x} = A(\vec{x}_h + \vec{x}_0) = A\vec{x}_h + A\vec{x}_0 = A\vec{x}_0$, since $\vec{x}_h$ is in $\ker(A)$.

b. If $\vec{x}_0$ and $\vec{x}_1$ are two solutions of the system $A\vec{x} = \vec{b}$, both from $(\ker A)^\perp$, then $\vec{x}_1 - \vec{x}_0$ is in the subspace $(\ker A)^\perp$ as well. Also, $A(\vec{x}_1 - \vec{x}_0) = A\vec{x}_1 - A\vec{x}_0 = \vec{b} - \vec{b} = \vec{0}$, so that $\vec{x}_1 - \vec{x}_0$ is in $\ker(A)$. By Fact 5.4.2c, it follows that $\vec{x}_1 - \vec{x}_0 = \vec{0}$, or $\vec{x}_1 = \vec{x}_0$, as claimed.

c. Write $\vec{x}_1 = \vec{x}_h + \vec{x}_0$ as in part a; note that $\vec{x}_h$ is orthogonal to $\vec{x}_0$. The claim now follows from the Pythagorean Theorem (Fact 5.1.8).

16. If A is an $m \times n$ matrix, then

$$\dim(\text{im}A)^{\perp} = m - \dim(\text{im}A) = m - \text{rank}(A)$$

$\uparrow$ Fact 5.4.2a $\qquad$ $\uparrow$ Fact 3.3.8

and $\dim(\ker(A^T)) = m - \text{rank}(A^T)$

$$\uparrow$$

Fact 3.3.5

It follows that $\text{rank}(A) = \text{rank}(A^T)$, as claimed.

17. Yes! By Fact 5.4.3, $\ker(A) = \ker(A^T A)$. Taking dimensions of both sides and using Fact 3.3.5, we find that $n - \text{rank}(A) = n - \text{rank}(A^T A)$; the claim follows.

20. Using Fact 5.4.7, we find $\vec{x}^* = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\vec{b} - A\vec{x}^* = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$.

Note that $\vec{b} - A\vec{x}^*$ is perpendicular to the two columns of A.

21. Using Fact 5.4.7, we find $\vec{x}^* = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\vec{b} - A\vec{x}^* = \begin{bmatrix} -12 \\ 36 \\ -18 \end{bmatrix}$, so that $\|\vec{b} - A\vec{x}^*\| = 42$.

22. Using Fact 5.4.7, we find $\vec{x}^* = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\vec{b} - A\vec{x}^* = \vec{0}$. This system is in fact consistent and $\vec{x}^*$ is the exact solution; the error $\|\vec{b} - A\vec{x}^*\|$ is 0.

23. Using Fact 5.4.7, we find $\vec{x}^* = \vec{0}$; here $\vec{b}$ is perpendicular to $\text{im}(A)$.

24. Using Fact 5.4.7, we find $\vec{x}^* = [2]$.

25. In this case, the normal equation $A^T A\vec{x} = A^T\vec{b}$ is $\begin{bmatrix} 5 & 15 \\ 15 & 45 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \end{bmatrix}$, which simplifies to $x_1 + 3x_2 = 1$, or $x_1 = 1 - 3x_2$. The solutions are of the form $\vec{x}^* = \begin{bmatrix} 1 - 3t \\ t \end{bmatrix}$, where t is an arbitrary constant.

26. Here, the normal equation $A^T A\vec{x} = A^T\vec{b}$ is $\begin{bmatrix} 66 & 78 & 90 \\ 78 & 93 & 108 \\ 90 & 108 & 126 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, with solutions $\vec{x}^* = \begin{bmatrix} t - \frac{7}{6} \\ 1 - 2t \\ t \end{bmatrix}$, where t is an arbitrary constant.

32. We want $\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$ of $f(t) = c_0 + c_1 t + c_2 t^2$ such that

$$\begin{aligned} 27 &= c_0 + 0c_1 + 0c_2 \\ 0 &= c_0 + 1c_1 + 1c_2 \\ 0 &= c_0 + 2c_1 + 4c_2 \\ 0 &= c_0 + 3c_1 + 9c_2 \end{aligned} \text{ or } \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 27 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

If we call the coefficient matrix A, we notice that $\ker(A) = \{\vec{0}\}$ so

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}^* = (A^T A)^{-1} A^T \begin{bmatrix} 27 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 25.65 \\ -28.35 \\ 6.75 \end{bmatrix} \text{ so } f^*(t) = 25.65 - 28.35t + 6.75t^2.$$

36. We want $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that

$$a + b\sin\left(\frac{2\pi}{365}28\right) + c\cos\left(\frac{2\pi}{365}28\right) = 10$$

$$a + b\sin\left(\frac{2\pi}{365}77\right) + c\cos\left(\frac{2\pi}{365}77\right) = 12$$

$$a + b\sin\left(\frac{2\pi}{365}124\right) + c\cos\left(\frac{2\pi}{365}124\right) = 14$$

$$a + b\sin\left(\frac{2\pi}{365}168\right) + c\cos\left(\frac{2\pi}{365}168\right) = 15.$$

Using $A = \begin{bmatrix} 1 & \sin\left(\frac{2\pi}{365}28\right) & \cos\left(\frac{2\pi}{365}28\right) \\ 1 & \sin\left(\frac{2\pi}{365}77\right) & \cos\left(\frac{2\pi}{365}77\right) \\ 1 & \sin\left(\frac{2\pi}{365}124\right) & \cos\left(\frac{2\pi}{365}124\right) \\ 1 & \sin\left(\frac{2\pi}{365}168\right) & \cos\left(\frac{2\pi}{365}168\right) \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 10 \\ 12 \\ 14 \\ 15 \end{bmatrix}$ we compute $\begin{bmatrix} a \\ b \\ c \end{bmatrix}^* = (A^T A)^{-1} A^T \vec{b} \approx$ $\begin{bmatrix} 12.25 \\ 0.394 \\ -2.726 \end{bmatrix}$ and $f^*(t) \approx 12.25 + 0.394\sin\left(\frac{2\pi}{365}t\right) - 2.726\cos\left(\frac{2\pi}{365}t\right)$.