3. Note that $\langle \vec{x}, \vec{y} \rangle = (S\vec{x})^T S\vec{y} = S\vec{x} \cdot S\vec{y}$.

a. We will check the four parts of Definition 5.5.1

α. $\langle \vec{x}, \vec{y} \rangle = S\vec{x} \cdot S\vec{y} = S\vec{y} \cdot S\vec{x} = \langle \vec{y}, \vec{x} \rangle$

β. $\langle \vec{x} + \vec{y}, \vec{z} \rangle = S(\vec{x} + \vec{y}) \cdot S\vec{z} = (S\vec{x} + S\vec{y}) \cdot S\vec{z} = (S\vec{x} \cdot S\vec{z}) + (S\vec{y} \cdot S\vec{z}) = \langle \vec{x}, \vec{z} \rangle + \langle \vec{y}, \vec{z} \rangle$

γ. $\langle c\vec{x}, \vec{y} \rangle = S(c\vec{x}) \cdot S\vec{y} = c(S\vec{x}) \cdot S\vec{y} = c\langle \vec{x}, \vec{y} \rangle$

δ. If $\vec{x} \neq 0$, then $\langle \vec{x}, \vec{x} \rangle = S\vec{x} \cdot S\vec{x} = \|S\vec{x}\|^2$ is positive if $S\vec{x} \neq \vec{0}$, that is, if $\vec{x}$ is not in the kernel of S. It is required that $S\vec{x} \neq \vec{0}$ whenever $\vec{x} \neq \vec{0}$, that is, $\ker(S) = \{\vec{0}\}$.

Answer: S must be invertible.

b. It is required that $\langle \vec{x}, \vec{y} \rangle = (S\vec{x})^T S\vec{y} = \vec{x}^T S^T S\vec{y}$ equal $\vec{x} \cdot \vec{y} = \vec{x}^T \vec{y}$ for all $\vec{x}$ and $\vec{y}$. This is the case if and only if $S^T S = I_n$, that is, S *is orthogonal.*

4. a. For column vectors $\vec{v}, \vec{w}$, we have $\langle \vec{v}, \vec{w} \rangle = \text{trace}(\vec{v}^T \vec{w}) = \text{trace}(\vec{v} \cdot \vec{w}) = \vec{v} \cdot \vec{w}$, the dot product.

b. For row vectors $\vec{v}, \vec{w}$, the ijth entry of $\vec{v}^T \vec{w}$ is $v_i w_j$, so that $\langle \vec{v}, \vec{w} \rangle = \text{trace}(\vec{v}^T \vec{w}) = \sum_{i=1}^{n} v_i w_i = \vec{v} \cdot \vec{w}$, again the dot product.

10. A function $g(t) = a + bt + ct^2$ is orthogonal to $f(t) = t$ if

$$\langle f, g\rangle = \int_{-1}^{1} (at + bt^2 + ct^3)\, dt = \left[\frac{a}{2}t^2 + \frac{b}{3}t^3 + \frac{c}{4}t^4\right]_{-1}^{1} = \frac{2}{3}b = 0,$$ that is, if $b = 0$.

Thus, the functions 1 and t^2 form a basis of the space of all functions in P_2 orthogonal to $f(t) = t$. To find an *orthonormal* basis $g_1(t), g_2(t)$, we apply Gram-Schmidt. Now $\|1\| = \frac{1}{2}\int_{-1}^{1} 1\, dt = 1$, so that we can let $g_1(t) = 1$. Then $g_2(t) = \dfrac{t^2 - \langle 1, t^2\rangle 1}{\|t^2 - \langle 1, t^2\rangle 1\|} = \dfrac{t^2 - \frac{1}{3}}{\|t^2 - \frac{1}{3}\|} = \dfrac{\sqrt{5}}{2}(3t^2 - 1)$

Answer: $g_1(t) = 1, g_2(t) = \dfrac{\sqrt{5}}{2}(3t^2 - 1)$

19. If we write $\vec{v} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \vec{w} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, and $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, then $\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle = [x_1 x_2] \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = px_1y_1 + qx_1y_2 + rx_2y_1 + sx_2y_2$. Note that in Exercise 15 we considered the special case $p = 1$. First it is required that $p = \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle$ be positive.

Now we can write $\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle = p \left[x_1y_1 + \frac{q}{p}x_1y_2 + \frac{r}{p}x_2y_1 + \frac{s}{p}x_2y_2 \right]$ and use our work in Exercise 15 $\left(\text{with } b = \frac{q}{p}, c = \frac{r}{p}, d = \frac{s}{p} \right)$ to see that the conditions $q = r$ and $q^2 < ps$ must hold. In summary, the function is an inner product if (and only if) the entries of matrix $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ satisfy the conditions $p > 0$, $q = r$ and $\det(A) = ps - q^2 > 0$.

20. a. $\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle = [x_1 x_2] \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = x_1 + 2x_2 = 0$ when $x_1 = -2x_2$. This is the line spanned by vector $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

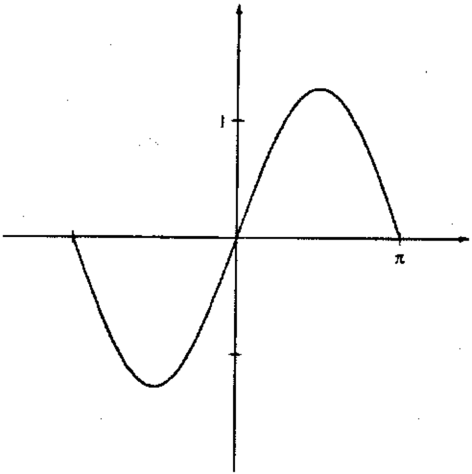
b. Since vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ are orthogonal, we merely have to multiply each of them with the reciprocal of its norm. Now $\left\| \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\|^2 = [1 \quad 0] \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1$, so that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a unit vector, and $\left\| \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\|^2 = [-2 \quad 1] \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = 4$, so that $\left\| \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\| = 2$. Thus $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix}$ is an orthonormal basis.

26. $a_0 = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t)\,dt = 0$

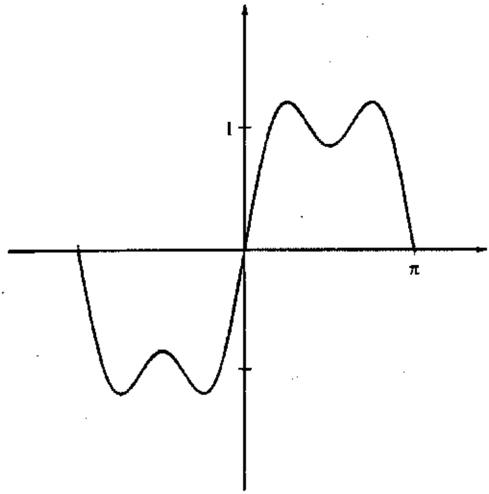
$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)\,dt = \frac{1}{\pi}\left\{-\int_{-\pi}^{0}\sin(kt)\,dt + \int_{0}^{\pi}\sin(kt)\,dt\right\} = \frac{2}{\pi}\int_{0}^{\pi}\sin(kt)\,dt$$

$$= -\frac{2}{k\pi}[\cos(kt)]_0^{\pi} = \begin{cases} 0 & \text{if } k \text{ is even} \\ \dfrac{4}{\pi k} & \text{if } k \text{ is odd} \end{cases}$$

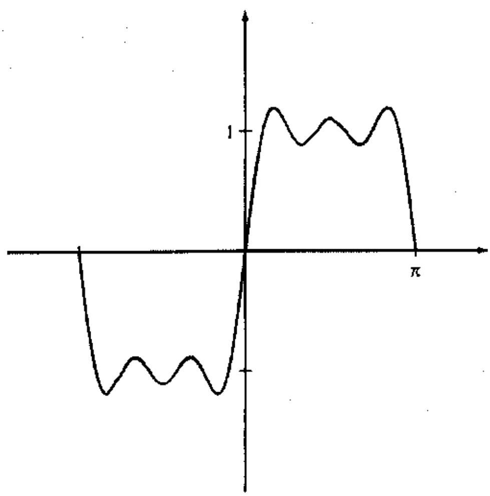
$c_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,dt = 0$, since the integrand is odd.



$$f_1(t) = f_2(t) = \frac{4}{\pi}\sin(t)$$



$$f_3(t) = f_4(t) = \frac{4}{\pi}\sin(t) + \frac{4}{3\pi}\sin(3t)$$



$$f_5(t) = f_6(t) = \frac{4}{\pi}\sin(t) + \frac{4}{3\pi}\sin(3t) + \frac{4}{5\pi}\sin(5t)$$

27. $a_0 = \frac{1}{\sqrt{2}\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\sqrt{2}}$.

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)\,dt = \frac{1}{\pi}\int_{0}^{\pi}\sin(kt)\,dt = -\frac{1}{k\pi}[\cos(kt)]_0^{\pi} = \begin{cases} 0 & \text{if } k \text{ is even} \\ \frac{2}{k\pi} & \text{if } k \text{ is odd} \end{cases}$$

$$c_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,dt = \frac{1}{\pi}\int_0^{\pi}\cos(kt)\,dt = \frac{1}{k\pi}[\sin(kt)]_0^{\pi} = 0$$

$$f_1(t) = f_2(t) = \frac{1}{2} + \frac{2}{\pi}\sin(t),\ f_3(t) = f_4(t) = \frac{1}{2} + \frac{2}{\pi}\sin(t) + \frac{2}{3\pi}\sin(3t)$$

$$\vdots$$

28. $\|f\|^2 = \langle f, f\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}(f(t))^2\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} 1\,dt = 2$

Now Fact 5.5.6 tells us that $\frac{16}{\pi^2} + \frac{16}{9\pi^2} + \frac{16}{25\pi^2} + \cdots = \frac{16}{\pi^2}\left(\sum_{k\text{ odd}}\frac{1}{k^2}\right) = 2$, or $\sum_{k\text{ odd}}\frac{1}{k^2} = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots = \frac{\pi^2}{8}$.