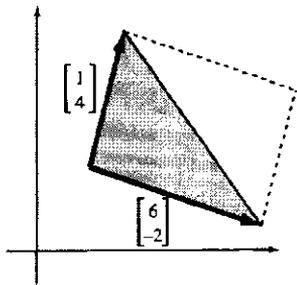
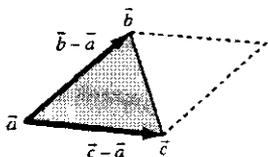


2. By Fact 6.3.3 Area =  $\frac{1}{2} \left| \det \begin{bmatrix} 3 & 8 \\ 7 & 2 \end{bmatrix} \right| = \frac{1}{2} |-50| = 25$

3. Area of triangle =  $\frac{1}{2} \left| \det \begin{bmatrix} 6 & 1 \\ -2 & 4 \end{bmatrix} \right| = 13$  (See figure.)



4. Note that area of triangle =  $\frac{1}{2} \left| \det \begin{bmatrix} b_1 - a_1 & c_1 - a_1 \\ b_2 - a_2 & c_2 - a_2 \end{bmatrix} \right|$ .

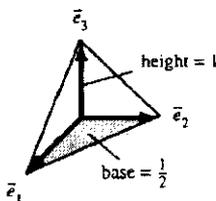


On the other hand,  $\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{bmatrix} = \det \begin{bmatrix} a_1 & b_1 - a_1 & c_1 - a_1 \\ a_2 & b_2 - a_2 & c_2 - a_2 \\ 1 & 0 & 0 \end{bmatrix} = \det \begin{bmatrix} b_1 - a_1 & c_1 - a_1 \\ b_2 - a_2 & c_2 - a_2 \end{bmatrix}$ .

$\uparrow$  subtract first column from second and third       $\uparrow$  expand down the first column

Therefore, area of triangle =  $\frac{1}{2} \left| \det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{bmatrix} \right|$ .

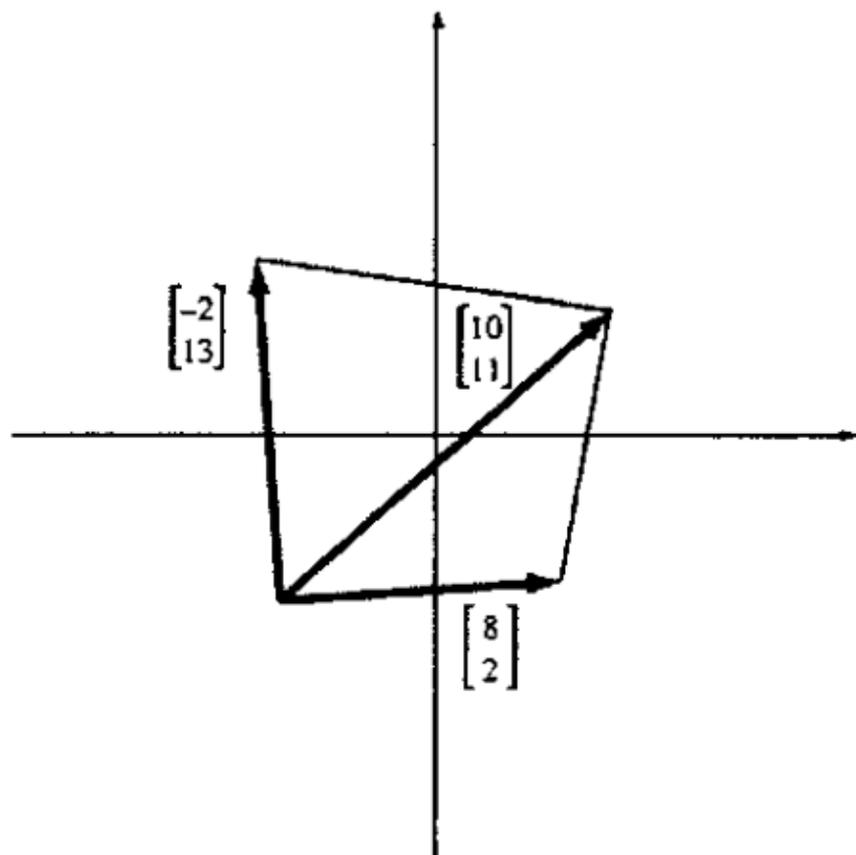
5. The volume of the tetrahedron  $T_0$  defined by  $\vec{e}_1, \vec{e}_2, \vec{e}_3$  is  $\frac{1}{3}(\text{base})(\text{height}) = \frac{1}{6}$  (formula for the volume of a pyramid).



The tetrahedron  $T$  defined by  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  can be obtained by applying the linear transformation with matrix  $[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$  to  $T_0$ .

Now we have  $\text{vol}(T) = |\det[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]| \text{vol}(T_0) = \frac{1}{6} |\det[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]| = \frac{1}{6} V(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ .

$$7. \text{ Area} = \frac{1}{2} \left| \det \begin{bmatrix} 10 & -2 \\ 11 & 13 \end{bmatrix} \right| + \frac{1}{2} \left| \det \begin{bmatrix} 8 & 10 \\ 2 & 11 \end{bmatrix} \right| = 110 \text{ (See Figure.)}$$

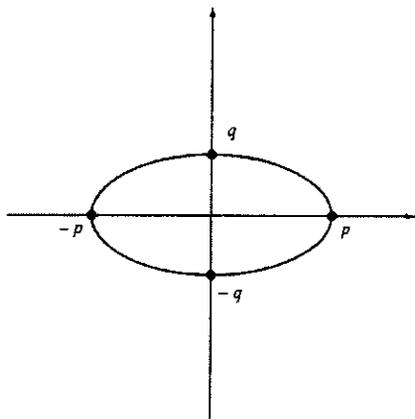


13. By Fact 6.3.7, the desired 2-volume is  $\sqrt{\det \left( \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \right)} = \sqrt{\det \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix}} = \sqrt{20}.$

14. By Fact 6.3.7, the desired 3-volume is

$$\sqrt{\det \left( \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 1 & 4 \end{bmatrix} \right)} = \sqrt{\det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 10 \\ 1 & 10 & 30 \end{bmatrix}} = \sqrt{6}.$$

18. a.

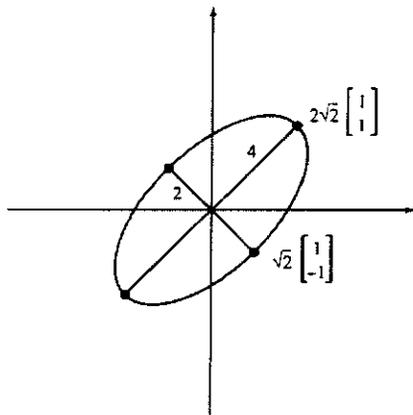


$$\begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix} \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} = \begin{bmatrix} p \cdot \cos(t) \\ q \cdot \sin(t) \end{bmatrix}, \text{ the ellipse with semi-axes } \pm \begin{bmatrix} p \\ 0 \end{bmatrix} \text{ and } \pm \begin{bmatrix} 0 \\ q \end{bmatrix}.$$

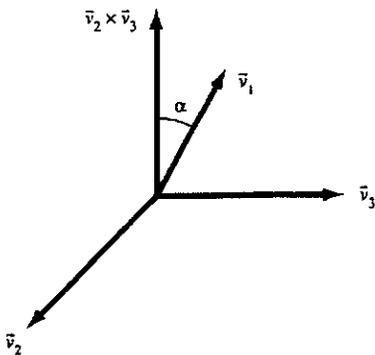
$$(\text{area of the ellipse}) = |\det(A)|(\text{area of the unit circle}) = pq\pi$$

b. By Fact 6.3.8,  $|\det(A)| = \frac{\text{area of the ellipse}}{\text{area of the unit circle}} = \frac{ab\pi}{\pi} = ab$  so  $|\det(A)| = ab$ .

c. The unit circle consists of all vectors of the form  $\vec{x} = \cos(t) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \sin(t) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ ; its image is the ellipse consisting of all vectors  $T(\vec{x}) = \cos(t) \underbrace{2\sqrt{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\text{semi-major axis}} + \sin(t) \underbrace{\sqrt{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\text{semi-minor axis}}$ .



19.  $\det[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3] = \vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3) = \|\vec{v}_1\| \|\vec{v}_2 \times \vec{v}_3\| \cos \alpha$  where  $\alpha$  is the angle between  $\vec{v}_1$  and  $\vec{v}_2 \times \vec{v}_3$  so  $\det[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3] > 0$  if and only if  $\cos \alpha > 0$ , i.e., if and only if  $\alpha$  is acute  $(0 \leq \alpha < \frac{\pi}{2})$ .



20. By Exercise 19,  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  constitute a positively oriented basis if and only if  $\det[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3] > 0$ . Assume that  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  is such a basis. We want to show that  $A\vec{v}_1, A\vec{v}_2, A\vec{v}_3$  is positively oriented if and only if  $\det(A) > 0$ . We have  $\det[A\vec{v}_1 \ A\vec{v}_2 \ A\vec{v}_3] = \det(A[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]) = \det(A) \det[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$  so since  $\det[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3] > 0$  by assumption,  $\det[A\vec{v}_1 \ A\vec{v}_2 \ A\vec{v}_3] > 0$  if and only if  $\det(A) > 0$ . Hence  $A$  is orientation preserving if and only if  $\det(A) > 0$ .

21. a. Reverses

Consider  $\vec{v}_2$  and  $\vec{v}_3$  in the plane (not parallel), and let  $\vec{v}_1 = \vec{v}_2 \times \vec{v}_3$ ; then  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  is a positively oriented basis, but  $T(\vec{v}_1) = -\vec{v}_1, T(\vec{v}_2) = \vec{v}_2, T(\vec{v}_3) = \vec{v}_3$  is negatively oriented.

b. Preserves

Consider  $\vec{v}_2$  and  $\vec{v}_3$  orthogonal to the line (not parallel), and let  $\vec{v}_1 = \vec{v}_2 \times \vec{v}_3$ ; then  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  is a positively oriented basis, and  $T(\vec{v}_1) = \vec{v}_1, T(\vec{v}_2) = -\vec{v}_2, T(\vec{v}_3) = -\vec{v}_3$  is positively oriented as well.

c. Reverses

The standard basis  $\vec{e}_1, \vec{e}_2, \vec{e}_3$  is positively oriented, but  $T(\vec{e}_1) = -\vec{e}_1, T(\vec{e}_2) = -\vec{e}_2, T(\vec{e}_3) = -\vec{e}_3$  is negatively oriented.

22. Here  $A = \begin{bmatrix} 3 & 7 \\ 4 & 11 \end{bmatrix}$ ,  $\det(A) = 5$ ,  $\vec{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ , so by Fact 6.3.9

$$x = \frac{\det \begin{bmatrix} 1 & 7 \\ 3 & 11 \end{bmatrix}}{5} = -2, \quad y = \frac{\det \begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix}}{5} = 1.$$

23. Here  $A = \begin{bmatrix} 5 & -3 \\ -6 & 7 \end{bmatrix}$ ,  $\det(A) = 17$ ,  $\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ , so by Fact 6.3.9

$$x_1 = \frac{\det \begin{bmatrix} 1 & -3 \\ 0 & 7 \end{bmatrix}}{17} = \frac{7}{17}, \quad x_2 = \frac{\det \begin{bmatrix} 5 & 1 \\ -6 & 0 \end{bmatrix}}{17} = \frac{6}{17}.$$

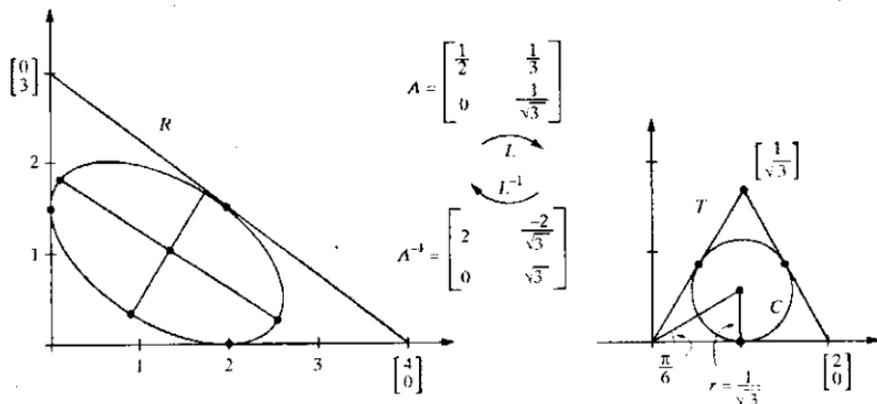
24. Here  $A = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 4 & 5 \\ 6 & 0 & 7 \end{bmatrix}$ ,  $\det(A) = 146$ ,  $\vec{b} = \begin{bmatrix} 8 \\ 3 \\ -1 \end{bmatrix}$ , so by Fact 6.3.9,

31. In what follows, we will freely use the fact that an invertible linear transformation  $L$  from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  maps an ellipse into an ellipse (see Exercise 52 of Section 2.2).

Now consider a linear transformation  $L$  that transforms our 3-4-5 right triangle  $R$  into an equilateral triangle  $T$ . If we place the vertices of the right triangle  $R$  at the points  $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$ ,  $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$ , and the origin, and the vertices of the equilateral triangle  $T$  at  $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$ , then the transformation  $L$  has the matrix

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{\sqrt{3}} \end{bmatrix}, \text{ with } \det(A) = \frac{1}{2\sqrt{3}}.$$

According to the hint,  $L$  will map the largest ellipse  $E$  inscribed into  $R$  into the circle  $C$  inscribed into  $T$ . The accompanying figure illustrates that the radius of  $C$  is  $\tan(\pi/6) = 1/\sqrt{3}$ , so that the area of  $C$  is  $\pi/3$ . Using the interpretation of the determinant as an expansion factor, we find that  $(\text{area of } C) = (\det A)(\text{area of } E)$ , or  $(\text{area of } E) = \frac{\text{area of } C}{\det(A)} = \frac{2\pi}{\sqrt{3}} \approx 3.6$



32. We will use the terminology introduced in the solution of Exercise 31 throughout. Note that the transformation  $L^{-1}$ , with matrix  $A^{-1} = \begin{bmatrix} 2 & -2/\sqrt{3} \\ 0 & \sqrt{3} \end{bmatrix}$ , maps the circle  $C$  (with radius  $1/\sqrt{3}$ ) into the ellipse  $E$ . Now consider a radial vector  $\vec{v} = \frac{1}{\sqrt{3}} \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$  of  $C$ , and find the maximal value  $M$  and the minimal value  $m$  of  $\|A^{-1}\vec{v}\|^2 = \frac{4}{3} + \frac{1}{9}\sin^2\theta - \frac{8}{3\sqrt{3}}(\sin\theta)(\cos\theta) = \frac{25}{18} - \frac{1}{18}(\cos 2\theta) - \frac{4}{3\sqrt{3}}(\sin 2\theta)$  (we are taking the square to facilitate the computations). Then  $\sqrt{M}$  and  $\sqrt{m}$  will be the lengths of the semi-axes of  $E$ . The function above is sinusoidal with average value  $\frac{25}{18}$  and amplitude  $\sqrt{\frac{1}{18^2} + \frac{16}{27}} = \frac{\sqrt{193}}{18}$ .

Thus  $M = \frac{25 + \sqrt{193}}{18}$  and  $m = \frac{25 - \sqrt{193}}{18}$ , so that the length of the semi-major axis of  $E$  is

$$\sqrt{M} = \sqrt{\frac{25 + \sqrt{193}}{18}} \approx 1.47, \text{ and for the semi-minor axis we get}$$

$$\sqrt{m} = \sqrt{\frac{25 - \sqrt{193}}{18}} \approx 0.79.$$