2. $\lambda_1 = 2, \lambda_2 = 0, E_2 = \text{span} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, E_0 = \text{span} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

   Eigenbasis: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

3. $\lambda_1 = 4, \lambda_2 = 9, E_4 = \text{span} \begin{bmatrix} 3 \\ -2 \end{bmatrix}, E_9 = \text{span} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

   Eigenbasis: $\begin{bmatrix} 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

4. $\lambda_1 = \lambda_2 = 1, E_1 = \text{span} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

   No eigenbasis

8. $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$, eigenbasis: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

9. $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = 0$, eigenbasis: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

10. $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = 0$, $E_1 = \text{span} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, E_0 = \text{span} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

No eigenbasis

11. $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = 3$, eigenbasis: $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

16. $\lambda_1 = 0$ (no other real eigenvalues), with eigenvector $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

No real eigenbasis

17. $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = \lambda_4 = 1$

with eigenbasis $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

18. $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = \lambda_4 = 1$, $E_0 = \text{span}(\vec{e}_1, \vec{e}_3)$, $E_1 = \text{span}(\vec{e}_2)$
No eigenbasis

21. We want $A$ such that $A\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$ and $A\begin{bmatrix}2\\3\end{bmatrix}=2\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}4\\6\end{bmatrix}$, i.e. $A\begin{bmatrix}1&2\\2&3\end{bmatrix}=\begin{bmatrix}1&4\\2&6\end{bmatrix}$ so

$$A=\begin{bmatrix}1&4\\2&6\end{bmatrix}\begin{bmatrix}1&2\\2&3\end{bmatrix}^{-1}=\begin{bmatrix}5&-2\\6&-2\end{bmatrix}.$$

The answer is unique.

27. By Fact 7.2.4, $f_A(\lambda) = \lambda^2 - 5\lambda + 6 = (\lambda - 3)(\lambda - 2)$ so $\lambda_1 = 2$, $\lambda_2 = 3$.

33. If $S^{-1}AS = B$, then
$S^{-1}(\lambda I_n - A)S = S^{-1}(\lambda S - AS) = \lambda S^{-1}S - S^{-1}AS = \lambda I_n - B.$

34. Note that $SB = AS$.

a. If $\vec{x}$ is in the kernel of $B$, then $AS\vec{x} = SB\vec{x} = S\vec{0} = \vec{0}$, so that $S\vec{x}$ is in $\ker(A)$.

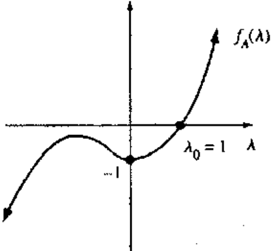
b. $T$ is clearly linear, and the transformation $R(\vec{x}) = S^{-1}\vec{x}$ is the inverse of $T$ (if $\vec{x}$ is in the kernel of $B$, then $S^{-1}\vec{x}$ is in the kernel of $A$, by part (a)).

c. The equation nullity($A$) = nullity($B$) follows from part (b); the equation rank($A$) = rank($B$) then follows from the rank-nullity theorem (Fact 3.3.9).

35. No, since the two matrices have different eigenvalues (see Fact 7.3.8.c).

36. No, since the two matrices have different traces (see Fact 7.3.8.d)

38. Note that $f_A(0) = \det(0I_3 - A) = \det(-A) = (-1)^3 \det(A) = -1$.
Since $\lim_{\lambda \to \infty} f_A(\lambda) = \infty$, the polynomial $f_A(\lambda)$ must have a positive root $\lambda_0$, by the Intermediate Value Theorem. In other words, the matrix $A$ will have a positive eigenvalue $\lambda_0$. Since $A$ is orthogonal, this eigenvalue $\lambda_0$ will be 1, by Fact 7.1.2. This means that there is a nonzero vector $\vec{v}$ in $\mathbb{R}^3$ such that $A\vec{v} = 1\vec{v} = \vec{v}$, as claimed.



41. The eigenvalues of $A$ are 1.2, $-0.8$, $-0.4$ with eigenvectors $\begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$.

Since $\vec{x}_0 = 50 \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix} + 50 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + 50 \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ we have $\vec{x}(t) = 50(1.2)^t \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix} + 50(-0.8)^t \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} +$ $50(-0.4)^t \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ so as $t$ goes to infinity $j(t) : n(t) : a(t)$ approaches the proportion $9 : 6 : 2$.

44. a. $a_{11} = 0.7$ means that only 70% of the pollutant present in Lake Silvaplana at a given time is still there a week later; some is carried down to Lake Sils by the river Inn, and some is absorbed or evaporates.

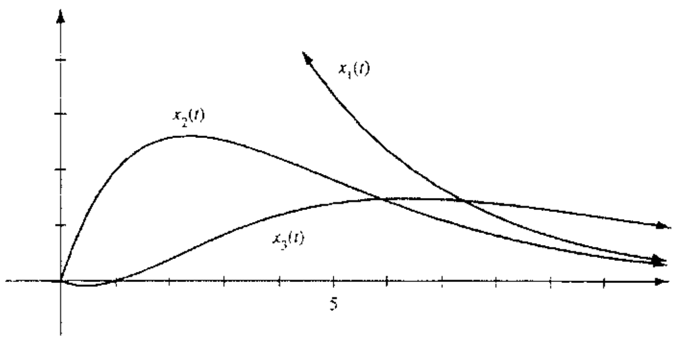
The other diagonal entries can be interpreted analogously. $a_{21} = 0.1$ means that 10% of the pollutant present in Lake Silvaplana at any given time can be found in Lake Sils a week later, carried down by the river Inn. The significance of the coefficient $a_{32} = 0.2$ is analogous; $a_{31} = 0$ means that no pollutant is carried down from Lake Silvaplana to Lake St. Moritz in just one week.

The matrix is lower triangular since no pollutant is carried from Lake Sils to Lake Silvaplana, for example (the river Inn flows the other way).

b. The eigenvalues of $A$ are 0.8, 0.6, 0.7 with corresponding eigenvectors $\begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\-2\end{bmatrix}$.

$\vec{x}(0) = \begin{bmatrix}100\\0\\0\end{bmatrix} = 100\begin{bmatrix}0\\0\\1\end{bmatrix} - 100\begin{bmatrix}0\\1\\-1\end{bmatrix} + 100\begin{bmatrix}1\\1\\-2\end{bmatrix}$ so $\vec{x}(t) = 100(0.8)^t\begin{bmatrix}0\\0\\1\end{bmatrix} - 100(0.6)^t\begin{bmatrix}0\\1\\-1\end{bmatrix} +$ $100(0.7)^t\begin{bmatrix}1\\1\\-2\end{bmatrix}$ or

$x_1(t) = 100(0.7)^t$
$x_2(t) = 100(0.7)^t - 100(0.6)^t$
$x_3(t) = 100(0.8)^t + 100(0.6)^t - 200(0.7)^t$



Using calculus, we find that the function $x_2(t) = 100(0.7)^t - 100(0.6)^t$ reaches its maximum at $t \approx 2.33$. Keep in mind, however, that our model holds for integer $t$ only.

48. a. We are told that

$a(t+1) = a(t) + j(t)$

$j(t+1) = a(t)$, so that $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

b. $f_A(\lambda) = \lambda(\lambda - 1) - 1 = \lambda^2 - \lambda - 1$ so $\lambda_{1,2} = \dfrac{1 \pm \sqrt{5}}{2}$ with eigenvectors $\begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$.

Since $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \dfrac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} - \dfrac{1}{\sqrt{5}} \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$ we have $\vec{x}(t) = \dfrac{1}{\sqrt{5}}(\lambda_1)^t \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} - \dfrac{1}{\sqrt{5}}(\lambda_2)^t \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$, i.e.

$a(t) = \dfrac{1}{\sqrt{5}}((\lambda_1)^{t+1} - (\lambda_2)^{t+1})$

$j(t) = \dfrac{1}{\sqrt{5}}((\lambda_1)^t - (\lambda_2)^t)$.

c. As $t \to \infty$, $\dfrac{a(t)}{j(t)} \to \lambda_1 = \dfrac{1 + \sqrt{5}}{2}$, since $|\lambda_2| < 1$.