9. Fails to be diagonalizable. There is only one eigenvalue, 1, with a one-dimensional eigenspace.

10. Fails to be diagonalizable. There is only one eigenvalue, 1, with a one-dimensional eigenspace.

11. Fails to be diagonalizable. The eigenvalues are 1,2,1, and the eigenspace $E_1 = \ker(I_3 - A) = \text{span}(\vec{e}_1)$ is only one-dimensional.

12. Diagonalizable. The eigenvalues are 2,1,1, with associated eigenvectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. If we let

$$S = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ then } S^{-1}AS = D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

15. Diagonalizable. The eigenvalues are $1, -1, 1$, with associated eigenvectors $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. If we let

$$S = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ then } S^{-1}AS = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

16. Diagonalizable. The eigenvalues are 3,2,1, with associated eigenvectors $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. If we let

$$S = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \text{ then } S^{-1}AS = D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

17. Diagonalizable. The eigenvalues are 0,3,0, with associated eigenvectors $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. If we let

$$S = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \text{ then } S^{-1}AS = D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

18. Diagonalizable. The eigenvalues are 3,2,1, with associated eigenvectors $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. If we let

$$S = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \text{ then } S^{-1}AS = D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

19. Fails to be diagonalizable. The eigenvalues are 1,0,1, and the eigenspace $E_1 = \ker(I_3 - A) = \operatorname{span}(\vec{e}_1)$ is only one-dimensional.

32. The eigenvalues of $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$ are 3 and 2, with associated eigenvectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. If we let $S = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, then $S^{-1}AS = D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$. Thus
$A = SDS^{-1}$ and $A^t = SD^tS^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^t & 0 \\ 0 & 2^t \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2(3^t) - 2^t & 2^{t+1} - 2(3^t) \\ 3^t - 2^t & 2^{t+1} - 3^t \end{bmatrix}$

33. The eigenvalues of $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ are 0 and 7, with associated eigenvectors $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$. If we let $S = \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix}$, then $S^{-1}AS = D = \begin{bmatrix} 0 & 0 \\ 0 & 7 \end{bmatrix}$. Thus $A = SDS^{-1}$ and $A^t = SD^tS^{-1} = \frac{1}{7}\begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 7^t \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 1 & 2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 7^t & 2(7^t) \\ 3(7^t) & 6(7^t) \end{bmatrix} = 7^{t-1}A$. We can find the same result more directly by observing that $A^2 = 7A$.

34. The eigenvalues of A are 1/4 and 1, with associated eigenvectors $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. If we let $S = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}$, then $S^{-1}AS = D = \begin{bmatrix} 1/4 & 0 \\ 0 & 1 \end{bmatrix}$. Thus $A = SDS^{-1}$ and $A^t = SD^tS^{-1} = \frac{1}{3}\begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} (1/4)^t & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 + 2(1/4)^t & 1 - (1/4)^t \\ 2 - 2(1/4)^t & 2 + (1/4)^t \end{bmatrix}$.

35. Matrix $\begin{bmatrix} -1 & 6 \\ -2 & 6 \end{bmatrix}$ has the eigenvalues 3 and 2. If $\vec{v}$ and $\vec{w}$ are associated eigenvectors, and if we let $S = [\vec{v}\,\vec{w}]$, then $S^{-1}\begin{bmatrix} -1 & 6 \\ -2 & 6 \end{bmatrix} S = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$, so that matrix $\begin{bmatrix} -1 & 6 \\ -2 & 6 \end{bmatrix}$ is indeed similar to $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$.

36. Yes. The matrices $\begin{bmatrix} -1 & 6 \\ -2 & 6 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$ both have the eigenvalues 3 and 2, so that each of them is similar to the diagonal matrix $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$, by Algorithm 7.4.4. Thus $\begin{bmatrix} -1 & 6 \\ -2 & 6 \end{bmatrix}$ is similar to $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$, by parts b and c of Fact 3.4.6.

47. The nonzero even functions, of the form $f(x) = a + cx^2$, are eigenfunctions with eigenvalue 1, and the nonzero odd functions, of the form $f(x) = bx$, have eigenvalue -1. Yes, T is diagonalizable, since the standard basis, $1, x, x^2$, is an eigenbasis for T.

48. Apply T to the standard basis: $T(1) = 1, T(x) = 2x$, and $T(x^2) = (2x)^2 = 4x^2$. This gives the eigenvalues 1, 2, and 4, with corresponding eigenfunctions $1, x, x^2$. Yes, T is diagonalizable, since the standard basis is an eigenbasis for T.

49. The matrix of T with respect to the standard basis $1, x, x^2$ is $B = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -6 \\ 0 & 0 & 9 \end{bmatrix}$. The eigenvalues of B are 1, 3, 9, with corresponding eigenvectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 4 \end{bmatrix}$. The eigenvalues of T are 1,3,9, with corresponding eigenfunctions $1, 2x - 1, 4x^2 - 4x + 1 = (2x - 1)^2$. Yes, T is diagonalizable, since the functions $1, 2x - 1, (2x - 1)^2$ from an eigenbasis.

50. The matrix of T with respect to the standard basis $1, x, x^2$ is $B = \begin{bmatrix} 1 & -3 & 9 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}$. The only eigenvalue of B is 1, with corresponding eigenvector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. The only eigenvalue of T is 1 as well, with corresponding eigenfunction $f(x) = 1$. T fails to be diagonalizable, since there is only one eigenvalue, with a one-dimensional eigenspace.

59. If $\vec{v}$ is an eigenvector with eigenvalue λ, then

$$f_A(A)\vec{v} = (A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I_n)\vec{v}$$
$$= \lambda^n \vec{v} + a_{n-1}\lambda^{n-1}\vec{v} + \cdots + a_1\lambda\vec{v} + a_0\vec{v}$$
$$= (\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0)\vec{v}$$
$$= f_A(\lambda)\vec{v} = 0\vec{v} = \vec{0}.$$

Since A is diagonalizable, any vector $\vec{x}$ in $\mathbb{R}^n$ can be written as a linear combination of eigenvectors, so that $f_A(A)\vec{x} = \vec{0}$. Since this equation holds for all $\vec{x}$ in $\mathbb{R}^n$, we have $f_A(A) = 0$, as claimed.

61. a. B is diagonalizable since it has three distinct eigenvalues, so that $S^{-1}BS$ is diagonal for some invertible S. But $S^{-1}AS = S^{-1}I_3S = I_3$ is diagonal as well. Thus A and B are indeed simultaneously diagonalizable.

b. There is an invertible S such that $S^{-1}AS = D_1$ and $S^{-1}BS = D_2$ are both diagonal. Then $A = SD_1S^{-1}$ and $B = SD_2S^{-1}$, so that $AB = (SD_1S^{-1})(SD_2S^{-1}) = SD_1D_2S^{-1}$ and $BA = (SD_2S^{-1})(SD_1S^{-1}) = SD_2D_1S^{-1}$. These two results agree, since $D_1D_2 = D_2D_1$ for the diagonal matrices D_1 and D_2.

c. Let A be I_n and B a nondiagonalizable $n \times n$ matrix, for example, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

d. Suppose $BD = DB$ for a diagonal D with distinct diagonal entries. The ijth entry of the matrix $BD = DB$ is $b_{ij}d_{jj} = d_{ii}b_{ij}$. For $i \neq j$ this implies that $b_{ij} = 0$. Thus B must be diagonal.

e. Since A has n distinct eigenvalues, A is diagonalizable, that is, there is an invertible S such that $S^{-1}AS = D$ is a diagonal matrix with n distinct diagonal entries. We claim that $S^{-1}BS$ is diagonal as well; by part d it suffices to show that $S^{-1}BS$ commutes with $D = S^{-1}AS$. This is easy to verify: $(S^{-1}BS)D = (S^{-1}BS)(S^{-1}AS) = S^{-1}BAS = S^{-1}ABS = (S^{-1}AS)(S^{-1}BS) = D(S^{-1}BS)$.