

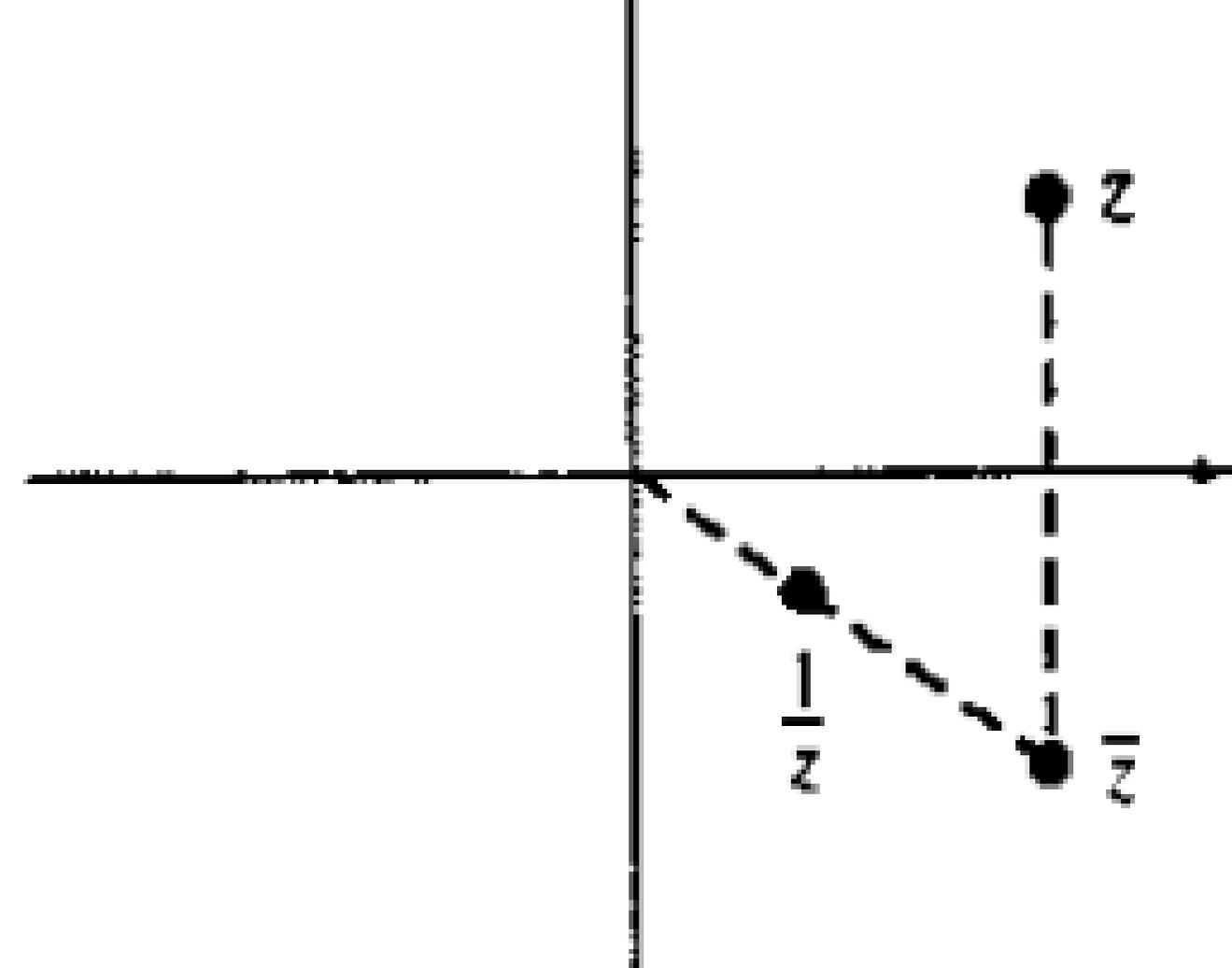
4. Let $z = r(\cos \phi + i \sin \phi)$ then $w = \sqrt{r} \left(\cos \left(\frac{\phi + 2\pi k}{2} \right) + i \sin \left(\frac{\phi + 2\pi k}{2} \right) \right)$, $k = 0, 1$.

5. Let $z = r(\cos \phi + i \sin \phi)$ then $w = \sqrt[n]{r} \left(\cos \left(\frac{\phi + 2\pi k}{n} \right) + i \sin \left(\frac{\phi + 2\pi k}{n} \right) \right)$, $k = 0, 1, 2, \dots, n - 1$.

6. If we have $z = r(\cos \phi + i \sin \phi)$ then $\frac{1}{z}$ must have the property that $z \cdot \frac{1}{z} = 1 = \cos 0 + i \sin 0$

i.e. $|z| \cdot \left| \frac{1}{z} \right| = 1$ and $\arg \left(z \cdot \frac{1}{z} \right) = \arg(z) + \arg \left(\frac{1}{z} \right) = 0$ so $\frac{1}{z} = \frac{1}{r} (\cos(-\phi) + i \sin(-\phi)) =$

$\frac{1}{r} (\cos \phi - i \sin \phi)$ (since cosine is even, sine odd). Hence $\frac{1}{z}$ is a real scalar multiple of \bar{z} .



11. Notice that $f(1) = 0$ so $\lambda = 1$ is a root of f . Hence $f(\lambda) = (\lambda - 1)g(\lambda)$, where $g(\lambda) = \frac{f(\lambda)}{\lambda - 1} = \lambda^2 - 2\lambda + 5$. Setting $g(\lambda) = 0$ we get $\lambda = 1 \pm 2i$ so that $f(\lambda) = (\lambda - 1)(\lambda - 1 - 2i)(\lambda - 1 + 2i)$.

12. We will use the facts:

i) $\overline{z + w} = \bar{z} + \bar{w}$ and

ii) $\overline{(z^n)} = \bar{z}^n$

These are easy to check. Assume λ_0 is a complex root of $f(\lambda) = a_n\lambda^n + \cdots + a_1\lambda + a_0$ where the coefficients a_j are real. Since λ_0 is a root of f , we have $a_n\lambda_0^n + a_{n-1}\lambda_0^{n-1} + \cdots + a_1\lambda_0 + a_0 = 0$.

Taking the conjugate of both sides we get $\overline{a_n \lambda_0^n + a_{n-1} \lambda_0^{n-1} + \dots + a_1 \lambda_0 + a_0} = \overline{0}$ so by fact i), and factoring the real constants we get $a_n \overline{\lambda_0^n} + a_{n-1} \overline{\lambda_0^{n-1}} + \dots + a_1 \overline{\lambda_0} + a_0 = 0$.

Now, by fact ii), $a_n (\overline{\lambda_0})^n + a_{n-1} (\overline{\lambda_0})^{n-1} + \dots + a_1 \overline{\lambda_0} + a_0 = 0$, i.e. $\overline{\lambda_0}$ is also a root of f , as claimed.

19. a. Since A has eigenvalues 1 and 0 associated with V and V^\perp respectively and since V is the eigenspace of $\lambda = 1$, by Fact 7.5.5, $\text{tr}(A) = m$, $\det(A) = 0$.

b. Since B has eigenvalues 1 and -1 associated with V and V^\perp respectively and since V is the eigenspace associated with $\lambda = 1$, $\text{tr}(A) = m - (n - m) = 2m - n$, $\det B = (-1)^{n-m}$.

20. $f_A(\lambda) = (\lambda - 3)(\lambda + 3) + 10 = \lambda^2 + 1$ so $\lambda_{1,2} = \pm i$.

21. $f_A(\lambda) = (\lambda - 11)(\lambda + 7) + 90 = \lambda^2 - 4\lambda + 13$ so $\lambda_{1,2} = 2 \pm 3i$.

24. $f_A(\lambda) = \lambda^3 - 3\lambda^2 + 7\lambda - 5$ so $\lambda_1 = 1, \lambda_{2,3} = 1 \pm 2i$. (See Exercise 11.)

25. $f_A(\lambda) = \lambda^4 - 1 = (\lambda^2 - 1)(\lambda^2 + 1) = (\lambda - 1)(\lambda + 1)(\lambda - i)(\lambda + i)$ so $\lambda_{1,2} = \pm 1$ and $\lambda_{3,4} = \pm i$

26. $f_A(\lambda) = (\lambda^2 - 2\lambda + 2)(\lambda^2 - 2\lambda) = (\lambda^2 - 2\lambda + 2)(\lambda - 2)\lambda = 0$, so $\lambda_{1,2} = 1 \pm i, \lambda_3 = 2, \lambda_4 = 0$.

27. By Fact 7.5.5 $\text{tr}(A) = \lambda_1 + \lambda_2 + \lambda_3$, $\det(A) = \lambda_1\lambda_2\lambda_3$ but $\lambda_1 = \lambda_2 \neq \lambda_3$ by assumption, so $\text{tr}(A) = 1 = 2\lambda_2 + \lambda_3$ and $\det(A) = 3 = \lambda_2^2\lambda_3$.

Solving for λ_2, λ_3 we get $-1, 3$ hence $\lambda_1 = \lambda_2 = -1$ and $\lambda_3 = 3$. (Note that the eigenvalues must be real; why?)

28. By Fact 7.5.5

$$2z\bar{z} = 50$$

$$2 + z + \bar{z} = 8$$

If $z = a + bi$ we get $(a + bi) + (a - bi) = 6$ so $a = 3$.

$(a + bi)(a - bi) = 25$, i.e. $a^2 + b^2 = 25$ or $9 + b^2 = 25$ so $b = \pm 4$.

Hence $\lambda_{2,3} = 3 \pm 4i$.

29. $\text{tr}(A) = 0$ so $\lambda_1 + \lambda_2 + \lambda_3 = 0$.

Also, we can compute $\det(A) = bcd > 0$ since $b, c, d > 0$. Therefore, $\lambda_1\lambda_2\lambda_3 > 0$.

Hence two of the eigenvalues must be negative, and the largest one (in absolute value) must be positive.

30. a. The i th entry of $A\vec{x}$ is $\sum_{k=1}^n a_{ik}x_k$, so that the sum of all the entries of $A\vec{x}$ is

$$\sum_{i=1}^n \sum_{k=1}^n a_{ik}x_k = \sum_{k=1}^n \sum_{i=1}^n a_{ik}x_k = \sum_{k=1}^n \left(\sum_{i=1}^n a_{ik} \right) x_k = \sum_{k=1}^n x_k = 1.$$

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b. As we do some computer experiments, A^t appears to approach a matrix with identical columns, with column sum 1. Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ be an eigenbasis with $\lambda_1 = 1$ and $|\lambda_j| < 1$ for $j = 2, \dots, n$. For a fixed i , write $\vec{e}_i = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$, so that (i th column of A^t) $= A^t\vec{e}_i = c_1\vec{v}_1 + [c_2\lambda_2^t\vec{v}_2 + \dots + c_n\lambda_n^t\vec{v}_n]$.

(The term in square brackets goes to zero as t goes to infinity.)

Therefore, $\lim_{t \rightarrow \infty} (\textit{i}th \textit{ column of } A^t) = \lim_{t \rightarrow \infty} (A^t\vec{e}_i) = c_1\vec{v}_1$.

Furthermore, the entries of $A^t\vec{e}_i$ add up to 1, for all t , by part a. Therefore, the same is true for the limit (since the limit of a sum is the sum of the limits).

It follows that $\lim_{t \rightarrow \infty} (A^t)$ exists and has identical columns, with column sum 1, as claimed.

31. No matter how we choose A , $\frac{1}{15}A$ is a regular transition matrix, so that $\lim_{t \rightarrow \infty} \left(\frac{1}{15}A \right)^t$ is a matrix with identical columns by Exercise 30. Therefore, the columns of A^t "become more and more alike" as t approaches infinity, in the sense that $\lim_{t \rightarrow \infty} \frac{ij\text{th entry of } A^t}{ik\text{th entry of } A^t} = 1$ for all i, j, k .

$$32. \text{ a. } \vec{x}(t) = \begin{bmatrix} a(t) \\ m(t) \\ s(t) \end{bmatrix} = \begin{bmatrix} 0.6a(t) + 0.1m(t) + 0.5s(t) \\ 0.2a(t) + 0.7m(t) + 0.1s(t) \\ 0.2a(t) + 0.2m(t) + 0.4s(t) \end{bmatrix} \text{ so } A = \begin{bmatrix} 0.6 & 0.1 & 0.5 \\ 0.2 & 0.7 & 0.1 \\ 0.2 & 0.2 & 0.4 \end{bmatrix}.$$

Note that A is a regular transition matrix.

b. By Exercise 30, $\lim_{t \rightarrow \infty} (A^t) = [\vec{v} \vec{v} \vec{v}]$, where \vec{v} is the unique eigenvector of A with eigenvalue 1 and

column sum 1. We find that $\vec{v} = \begin{bmatrix} 0.4 \\ 0.35 \\ 0.25 \end{bmatrix}$.

Now $\lim_{t \rightarrow \infty} \vec{x}(t) = \lim_{t \rightarrow \infty} (A^t \vec{x}_0) = \left(\lim_{t \rightarrow \infty} A^t \right) \vec{x}_0 = [\vec{v} \vec{v} \vec{v}] \vec{x}_0 = \vec{v}$, since the components of \vec{x}_0 add up to 1.

The market shares approach 40%, 35%, and 25%, respectively, regardless of the initial shares.

36. a. The entries in the first row are age-specific birth rates and the entries just below the diagonal are age-specific survival rates. For example, the entry 1.6 in the first row tells us that during the next 15 years the people who are 15–30 years old today will *on average* have 1.6 children (3.2 per couple) who will survive to the next census. The entry 0.53 tells us that 53% of those in the age group 45–60 today will still be alive in 15 years (they will then be in the age group 60–75).
- b. Using technology, we find the largest eigenvalue $\lambda_1 = 1.908$ with associated eigenvector

$$\vec{v}_1 \approx \begin{bmatrix} 0.574 \\ 0.247 \\ 0.115 \\ 0.047 \\ 0.014 \\ 0.002 \end{bmatrix}.$$

The components of \vec{v}_1 give the distribution of the population among the age groups in the long run, assuming that current trends continue. λ_1 gives the factor by which the population will grow in the long run in a period of 15 years; this translates to an annual growth factor of $\sqrt[15]{1.908} \approx 1.044$, or an annual growth of about 4.4%.