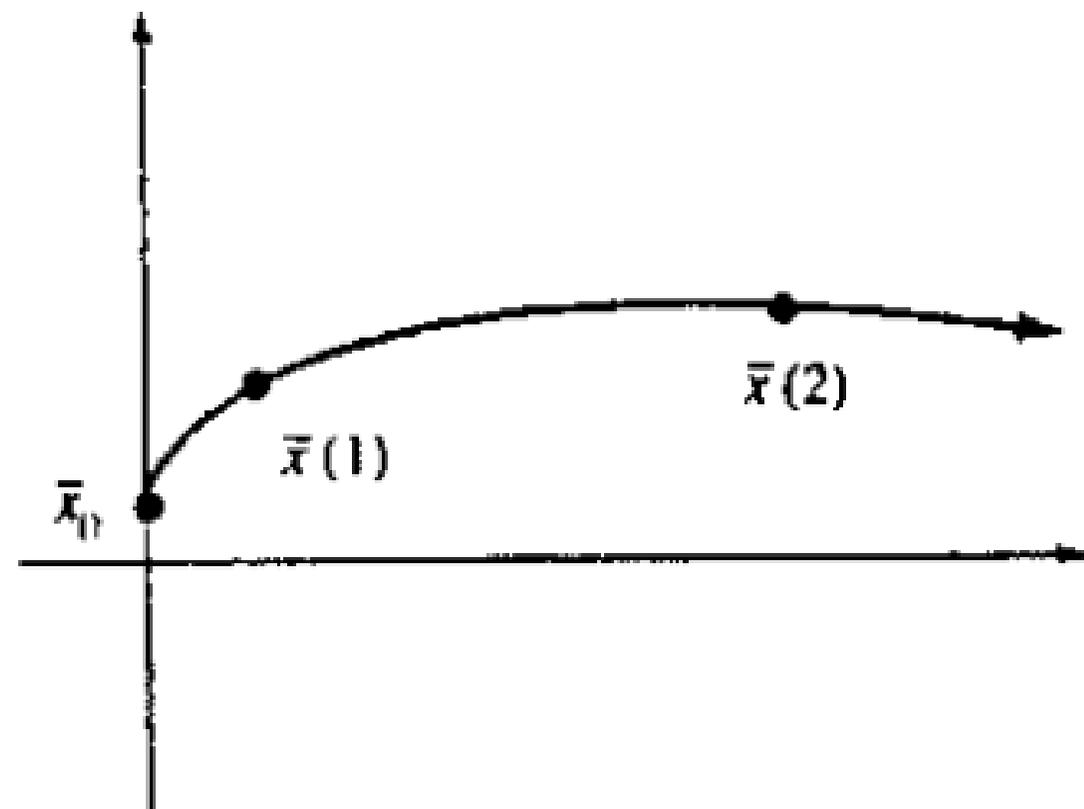


7. $\lambda_{1,2} = 0.9 \pm (0.5)i$ so $|\lambda_1| = |\lambda_2| = \sqrt{0.81 + 0.25} > 1$ and $\vec{0}$ is not a stable equilibrium.
8. $\lambda_1 = 0.9, \lambda_2 = 0.8$ so $\vec{0}$ is a stable equilibrium.
9. $\lambda_{1,2} = 0.8 \pm (0.6)i, \lambda_3 = 0.7$, so $|\lambda_1| = |\lambda_2| = 1$ and $\vec{0}$ is not a stable equilibrium.
10. $\lambda_{1,2} = 0, \lambda_3 = 0.9$ so $\vec{0}$ is a stable equilibrium.

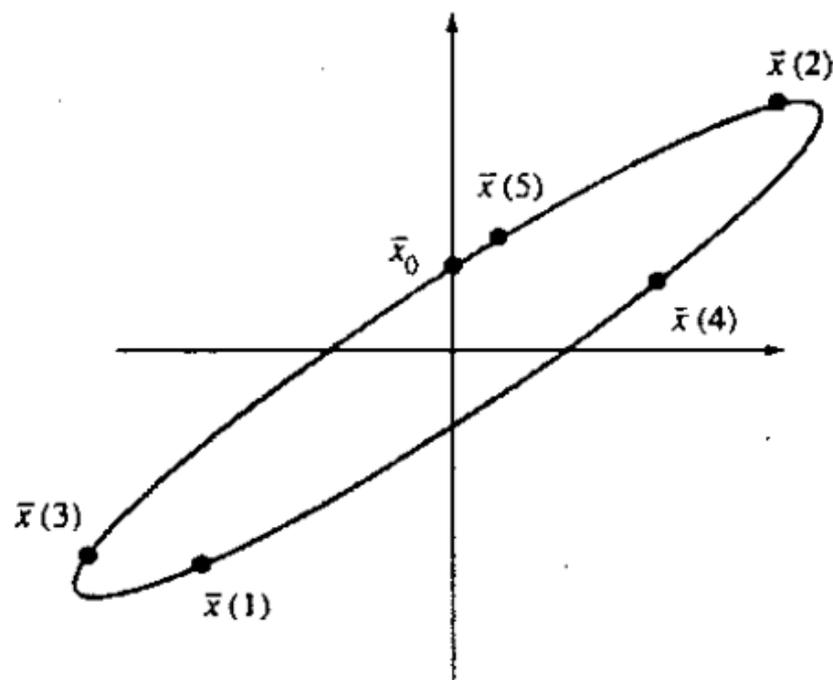
$$20. \lambda_{1,2} = 4 \pm 3i, r = 5, \phi = \arctan\left(\frac{3}{4}\right) \approx 0.64, \text{ so } \lambda_1 \approx 5(\cos(0.64) + i \sin(0.64)), [\vec{w} \ \vec{v}] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \vec{x}(t) \approx 5^t \begin{bmatrix} \sin(0.64t) \\ \cos(0.64t) \end{bmatrix}.$$

Spirals outwards (rotation-dilation).



24. $\lambda_{1,2} = -0.8 \pm 0.6i$, $r = 1$, $\phi = \pi - \arctan\left(\frac{.6}{.8}\right) \approx 2.5$ (second quadrant)

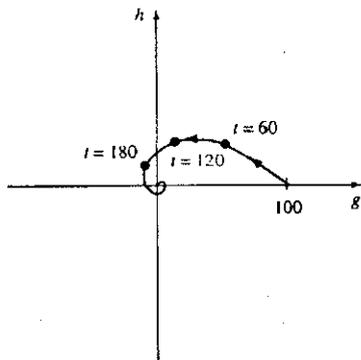
$[\vec{w} \ \vec{v}] = \begin{bmatrix} 0 & -5 \\ 1 & -3 \end{bmatrix}$, $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ so $\vec{x}(t) = \begin{bmatrix} -5 \sin(\phi t) \\ \cos(\phi t) - 3 \sin(\phi t) \end{bmatrix}$, an ellipse.



32. $\lambda_{1,2} = 0.99 \pm (0.01)i$ with eigenvector $\begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for λ_1 , so $\vec{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\phi = \arctan\left(\frac{0.01}{0.99}\right) \approx$

0.01 , $r \approx 0.99$. Hence $\vec{x}(0) = \begin{bmatrix} 100 \\ 0 \end{bmatrix} = 100\vec{w} + 0\vec{v}$ and

$$\vec{x}(t) \approx 0.99^t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(0.01t) & -\sin(0.01t) \\ \sin(0.01t) & \cos(0.01t) \end{bmatrix} \begin{bmatrix} 100 \\ 0 \end{bmatrix} = 0.99^t \cdot 100 \begin{bmatrix} \cos(0.01t) \\ \sin(0.01t) \end{bmatrix} \text{ i.e. } \vec{x}(t) \text{ spirals in.}$$



Hence, both glucose and excess insulin oscillate with damped oscillations until all of the excess sugar has been eliminated.

34. a. If $|\det A| = |\lambda_1 \lambda_2 \cdots \lambda_n| = |\lambda_1 \lambda_2| \cdots |\lambda_n| > 1$ then at least one eigenvalue is greater than one in modulus and the zero state fails to be stable.

b. If $|\det A| = |\lambda_1| |\lambda_2| \cdots |\lambda_n| < 1$ we cannot conclude anything about the stability of $\vec{0}$.

$|2\|0.1| < 1$ and $|0.2\|0.1| < 1$ but in the first case we would not have stability, in the second case we would.

35. a. Let $\vec{v}_1, \dots, \vec{v}_n$ be an eigenbasis for A . Then $\vec{x}(t) = \sum_{i=1}^n c_i \lambda_i^t \vec{v}_i$ and

$$\|\vec{x}(t)\| = \left\| \sum_{i=1}^n c_i \lambda_i^t \vec{v}_i \right\| \leq \sum_{i=1}^n \|c_i \lambda_i^t \vec{v}_i\| = \sum_{i=1}^n |\lambda_i|^t \|c_i \vec{v}_i\| \leq \sum_{i=1}^n \|c_i \vec{v}_i\|.$$

\uparrow
 ≤ 1

The last quantity, $\sum_{i=1}^n \|c_i \vec{v}_i\|$, gives the desired bound M .

b. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ represents a shear parallel to the x -axis, with $A \begin{bmatrix} k \\ 1 \end{bmatrix} = \begin{bmatrix} k+1 \\ 1 \end{bmatrix}$, so that $\vec{x}(t) = A^t \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} t \\ 1 \end{bmatrix}$ is not bounded. This does not contradict part a, since there is no eigenbasis for A .

38. a. $T(\vec{v}) = A\vec{v} + \vec{b} = \vec{v}$ if $\vec{v} - A\vec{v} = \vec{b}$ or $(I_n - A)\vec{v} = \vec{b}$.

$I_n - A$ is invertible since 1 is not an eigenvalue of A . Therefore, $\vec{v} = (I_n - A)^{-1} \vec{b}$ is the only solution.

b. Let $\vec{y}(t) = \vec{x}(t) - \vec{v}$ be the deviation of $\vec{x}(t)$ from the equilibrium \vec{v} .

Then $\vec{y}(t+1) = \vec{x}(t+1) - \vec{v} = A\vec{x}(t) + \vec{b} - \vec{v} = A(\vec{y}(t) + \vec{v}) + \vec{b} - \vec{v} = A\vec{y}(t) + A\vec{v} + \vec{b} - \vec{v} = A\vec{y}(t)$,
so that $\vec{y}(t) = A^t \vec{y}(0)$, or $\vec{x}(t) = \vec{v} + A^t (\vec{x}_0 - \vec{v})$.

$\lim_{t \rightarrow \infty} \vec{x}(t) = \vec{v}$ for all \vec{x}_0 if $\lim_{t \rightarrow \infty} A^t (\vec{x}_0 - \vec{v}) = \vec{0}$. This is the case if the modulus of all the eigenvalues of A is less than 1.

40. Note that A can be partitioned as $A = \begin{bmatrix} B & -C^T \\ C & B^T \end{bmatrix}$, where B and C are rotation-dilation matrices.

Also note that $BC = CB$, $B^T B = (p^2 + q^2)I_2$, and $C^T C = (r^2 + s^2)I_2$.

a. $A^T A = \begin{bmatrix} B^T & C^T \\ -C & B \end{bmatrix} \begin{bmatrix} B & -C^T \\ C & B^T \end{bmatrix} = (p^2 + q^2 + r^2 + s^2)I_4$

b. By part a, $A^{-1} = \frac{1}{p^2 + q^2 + r^2 + s^2} A^T$ if $A \neq 0$.

c. $(\det A)^2 = (p^2 + q^2 + r^2 + s^2)^4$, by part a, so that $\det A = \pm(p^2 + q^2 + r^2 + s^2)^2$.

The diagonal pattern makes the contribution $+p^4$, so that $\det A = (p^2 + q^2 + r^2 + s^2)^2$.

d. Consider $\det(\lambda I_4 - A)$. Note that the matrix $\lambda I_4 - A$ has the same "format" as A , with p replaced by $\lambda - p$ and q, r, s by $-q, -r, -s$, respectively. By part c, $\det(\lambda I_4 - A) = ((\lambda - p)^2 + q^2 + r^2 + s^2)^2 = 0$ when

$$(\lambda - p)^2 = -q^2 - r^2 - s^2$$

$$\lambda - p = \pm i\sqrt{q^2 + r^2 + s^2}$$

$$\lambda_{1,2} = p \pm i\sqrt{q^2 + r^2 + s^2}$$

Each of these eigenvalues has algebraic multiplicity 2 (if $q = r = s = 0$ then $\lambda = p$ has algebraic multiplicity 4).

e. By part a we can write $A = \underbrace{\sqrt{p^2 + q^2 + r^2 + s^2}}_S \left(\frac{1}{\sqrt{p^2 + q^2 + r^2 + s^2}} A \right)$, where S is orthogonal.

Therefore, $\|A\vec{x}\| = \|\sqrt{p^2 + q^2 + r^2 + s^2}(S\vec{x})\| = \sqrt{p^2 + q^2 + r^2 + s^2}\|\vec{x}\|$.

f. Let $A = \begin{bmatrix} 3 & -3 & -4 & -5 \\ 3 & 3 & 5 & -4 \\ 4 & -5 & 3 & 3 \\ 5 & 4 & -3 & 3 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 4 \end{bmatrix}$; then $A\vec{x} = \begin{bmatrix} -39 \\ 13 \\ 18 \\ 13 \end{bmatrix}$.

By part e, $\|A\vec{x}\|^2 = (3^2 + 3^2 + 4^2 + 5^2)\|\vec{x}\|^2$, or

$$39^2 + 13^2 + 18^2 + 13^2 = (3^2 + 3^2 + 4^2 + 5^2)(1^2 + 2^2 + 4^2 + 4^2), \text{ as desired.}$$

g. Any positive integer m can be written as $m = p_1 p_2 \dots p_n$. Using part f repeatedly we see that the numbers $p_1, p_1 p_2, p_1 p_2 p_3, \dots, p_1 p_2 p_3 \dots p_{n-1}$, and finally $m = p_1 \dots p_n$ can be expressed as the sums of four squares.