3. $\frac{1}{\sqrt{5}}\begin{bmatrix}2\\1\end{bmatrix}, \frac{1}{\sqrt{5}}\begin{bmatrix}-1\\2\end{bmatrix}$ is an orthonormal eigenbasis.

4. $\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\-1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\2\end{bmatrix}$ is an orthonormal eigenbasis.

5. Eigenvalues -1, -1, 2

Choose $\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix}$ in E_{-1} and $\vec{v}_2 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$ in E_2 and let $\vec{v}_3 = \vec{v}_1 \times \vec{v}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\-2\end{bmatrix}$.

6. $\frac{1}{3}\begin{bmatrix}2\\2\\1\end{bmatrix}, \frac{1}{3}\begin{bmatrix}2\\-1\\-2\end{bmatrix}, \frac{1}{3}\begin{bmatrix}1\\-2\\2\end{bmatrix}$ is an orthonormal eigenbasis.

10. $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 9$.

$\vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ is in E_0 and $\vec{v}_2 = \frac{1}{3} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ is in E_9.

Let $\vec{v}_3 = \vec{v}_1 \times \vec{v}_2 = \frac{1}{3\sqrt{5}} \begin{bmatrix} 2 \\ -4 \\ -5 \end{bmatrix}$; then $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ is an orthonormal eigenbasis.

$$S = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{3} & \frac{2}{3\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{3} & -\frac{4}{3\sqrt{5}} \\ 0 & \frac{2}{3} & -\frac{\sqrt{5}}{3} \end{bmatrix} \text{ and } D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

11. $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$ is an orthonormal eigenbasis, with $\lambda_1 = 2$, $\lambda_2 = 0$, and $\lambda_3 = 1$, so

$$S = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & 1 & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

12. a. $E_1 = \operatorname{span}\begin{bmatrix}1\\0\\2\end{bmatrix}$ and $E_{-1} = (E_1)^{\perp}$. An orthonormal eigenbasis is $\frac{1}{\sqrt{5}}\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \frac{1}{\sqrt{5}}\begin{bmatrix}2\\0\\-1\end{bmatrix}$.

b. Use Fact 7.4.1: $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

c. $A = SBS^{-1} = \begin{bmatrix} -0.6 & 0 & 0.8 \\ 0 & -1 & 0 \\ 0.8 & 0 & 0.6 \end{bmatrix}$, where $S = \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \\ \frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \end{bmatrix}$.

15. Yes, if $A\vec{v} = \lambda\vec{v}$, then $A^{-1}\vec{v} = \frac{1}{\lambda}\vec{v}$, so that an orthonormal eigenbasis for A is also an orthonormal eigenbasis for A^{-1} (with reciprocal eigenvalues).

16. a. $\ker(A)$ is four-dimensional, so that the eigenvalue 0 has multiplicity 4, and the remaining eigenvalue is $\mathrm{tr}(A) = 5$.

b. $B = A + 2I_5$, so that the eigenvalues are 2, 2, 2, 2, 7.

c. $\det(B) = 2^4 \cdot 7 = 112$ (product of eigenvalues)

17. If A is the $n \times n$ matrix with all 1's, then the eigenvalues of A are 0 (with multiplicity $n-1$) and n. Now $B = qA + (p-q)I_n$, so that the eigenvalues of B are $p - q$ (with multiplicity $n - 1$) and $qn + p - q$. Thus $\det(B) = (p-q)^{n-1}(qn + p - q)$.

18. By Fact 6.3.7, the volume is $|\det A| = \sqrt{\det(A^T A)}$. Now $\vec{v}_i \cdot \vec{v}_j = \|\vec{v}_i\|\|\vec{v}_j\| \cdot \cos(\alpha) = \frac{1}{2}$, so that $A^T A$ has all 1's on the diagonal and $\frac{1}{2}$'s outside. By Exercise 17 (with $p = 1$ and $q = \frac{1}{2}$), $\det(A^T A) = \left(\frac{1}{2}\right)^{n-1}\left(\frac{1}{2}n + \frac{1}{2}\right) = \left(\frac{1}{2}\right)^n (n+1)$, so that the volume is $\sqrt{\det(A^T A)} = \left(\frac{1}{2}\right)^{n/2}\sqrt{n+1}$.

19. Let $L(\vec{x}) = A\vec{x}$. Then $A^T A$ is symmetric, since $(A^T A)^T = A^T (A^T)^T = A^T A$, so that there is an orthonormal eigenbasis $\vec{v}_1, \ldots, \vec{v}_n$ for $A^T A$. Then the vectors $A\vec{v}_1, \ldots, A\vec{v}_n$ are orthogonal, since $A\vec{v}_i \cdot A\vec{v}_j = (A\vec{v}_i)^T A\vec{v}_j = \vec{v}_i^T A^T A\vec{v}_j = \vec{v}_i \cdot (A^T A\vec{v}_j) = \vec{v}_i \cdot (\lambda_j \vec{v}_j) = \lambda_j(\vec{v}_i \cdot \vec{v}_j) = 0$ if $i \neq j$.

20. By Exercise 19, there is an orthonormal basis $\vec{v}_1, \ldots, \vec{v}_n$ of $\mathbb{R}^n$ such that $T(\vec{v}_1), \ldots, T(\vec{v}_n)$ are orthogonal. Suppose that $T(\vec{v}_1), \ldots, T(\vec{v}_r)$ are nonzero and $T(\vec{v}_{r+1}), \ldots, T(\vec{v}_n)$ are zero. Then let $\vec{w}_i = \frac{1}{\|T(\vec{v}_i)\|}T(\vec{v}_i)$ for $i = 1, \ldots, r$ and choose an orthonormal basis $\vec{w}_{r+1}, \ldots, \vec{w}_m$ of $[\mathrm{span}(\vec{w}_1, \ldots, \vec{w}_r)]^\perp$. Then $\vec{w}_1, \ldots, \vec{w}_m$ does the job.

29. By Fact 5.4.1 $(\text{im } A)^{\perp} = \ker(A^T) = \ker(A)$, so that $\vec{v}$ is orthogonal to $\vec{w}$.

31. True; A is diagonalizable, that is, A is similar to a diagonal matrix D; then A^2 is similar to D^2. Now $\text{rank}(D) = \text{rank}(D^2)$ is the number of nonzero entries on the diagonal of D (and D^2). Since similar matrices have the same rank (by Fact 7.3.8b) we can conclude that $\text{rank}(A) = \text{rank}(D) = \text{rank}(D^2) = \text{rank}(A^2)$.

36. If $\vec{v}$ is an eigenvector with eigenvalue λ, then $\lambda\vec{v} = A\vec{v} = A^2\vec{v} = \lambda^2\vec{v}$, so that $\lambda = \lambda^2$ and therefore $\lambda = 0$ or $\lambda = 1$. Since A is symmetric, E_0 and E_1 are orthogonal complements, so that A represents the orthogonal projection onto E_1.