8.2

1. $A = \begin{bmatrix} 6 & -\frac{7}{2} \\ -\frac{7}{2} & 8 \end{bmatrix}$

2. $A = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$

3. $A = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 4 & \frac{7}{2} \\ 3 & \frac{7}{2} & 5 \end{bmatrix}$

4. $A = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$, positive definite

5. $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, indefinite (since $\det(A) < 0$)

6. $A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$, indefinite (since $\det(A) < 0$)

7. $A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 0 \end{bmatrix}$, indefinite (eigenvalues 2, -2, 3)

8. If $S^{-1}AS = D$ is diagonal, then $S^{-1}A^2S = D^2$, so that all eigenvalues of A^2 are ≥ 0. So A^2 is positive semi-definite; it is positive definite if and only if A is invertible.

9. a. $(A^2)^T = (A^T)^2 = (-A)^2 = A^2$, so that A^2 is symmetric.

   b. $q(\vec{x}) = \vec{x}^T A^2 \vec{x} = \vec{x}^T AA\vec{x} = -\vec{x}^T A^T A\vec{x} = -(A\vec{x}) \cdot (A\vec{x}) = -\|A\vec{x}\|^2 \leq 0$ for all $\vec{x}$, so that A^2 is negative semi-definite.

   c. If $\vec{v}$ is a complex eigenvector of A with eigenvalue λ, then $A^2\vec{v} = \lambda^2\vec{v}$, and $\lambda^2 \leq 0$, by part b. Therefore, λ is *imaginary*, that is, $\lambda = bi$ for a real b. Thus, the zero matrix is the only skew-symmetric matrix that is diagonalizable over $\mathbb{R}$.

10. $L(\vec{x}) = (\vec{x}+\vec{v})^T A(\vec{x}+\vec{v}) - \vec{x}^T A\vec{x} - \vec{v}^T A\vec{v} = \vec{x}^T A\vec{x} + \vec{x}^T A\vec{v} + \vec{v}^T A\vec{x} + \vec{v}^T A\vec{v} - \vec{x}^T A\vec{x} - \vec{v}^T A\vec{v} = \vec{x}^T A\vec{v} + \vec{v}^T A\vec{x} = \vec{v}^T A\vec{x} + \vec{v}^T A\vec{x} = (2\vec{v}^T A)\vec{x}$,

    ↑

    note that $\vec{x}^T A\vec{v}$ is a scalar so that $\vec{x}^T A\vec{v} = (\vec{x}^T A\vec{v})^T = \vec{v}^T A^T\vec{x} = \vec{v}^T A\vec{x}$ if A is symmetric.

    So L is linear with matrix $2\vec{v}^T A$.

11. The eigenvalues of A^{-1} are the reciprocals of those of A, so that A and A^{-1} have the same definiteness.

12. $\det(A)$ is the product of the two (real) eigenvalues. q is indefinite if an only if those have different signs, that is, their product is negative.

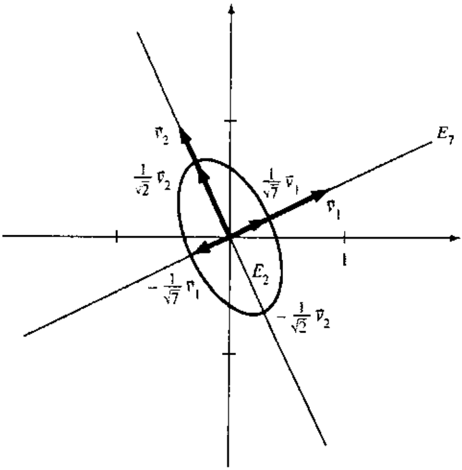
13. $q(\vec{e}_i) = \vec{e}_i \cdot A\vec{e}_i = a_{ii} > 0$

14. If $\det(A)$ is positive then both eigenvalues have the same sign, so that A is positive definite or negative definite. Since $\vec{e}_1 \cdot A\vec{e}_1 = a > 0$, A is in fact positive definite.

15. $A = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$; eigenvalues $\lambda_1 = 7$ and $\lambda_2 = 2$

    orthonormal eigenbasis $\vec{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

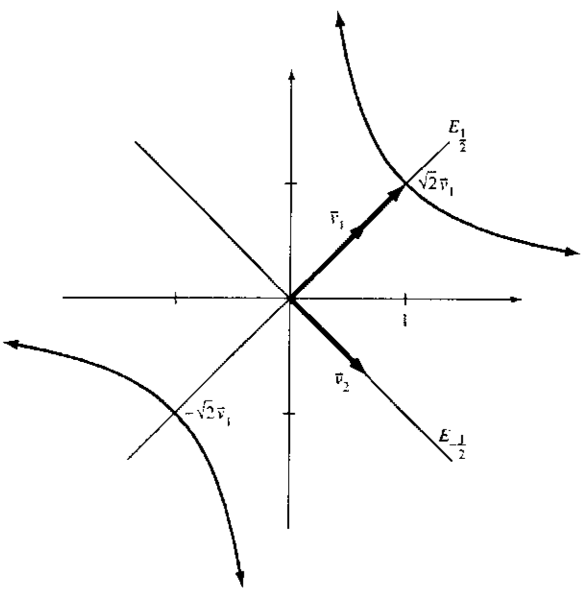
    $\lambda_1 c_1^2 + \lambda_2 c_2^2 = 1$ or $7c_1^2 + 2c_2^2 = 1$



16. $A = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$; eigenvalues $\lambda_1 = \dfrac{1}{2}$, and $\lambda_2 = -\dfrac{1}{2}$

orthonormal eigenbasis $\vec{v}_1 = \dfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \dfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

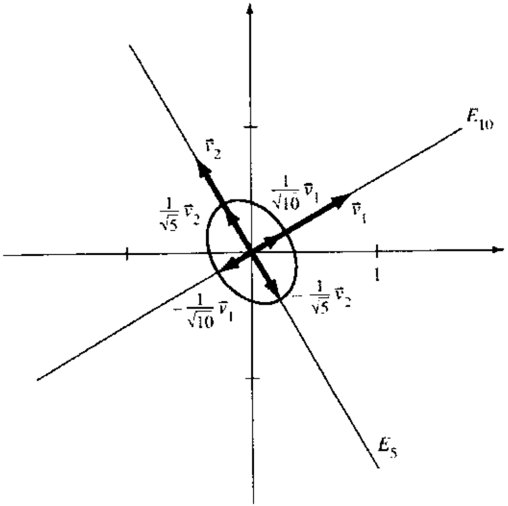
$\dfrac{1}{2}c_1^2 - \dfrac{1}{2}c_2^2 = 1$



18. $A = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}$, eigenvalues $\lambda_1 = 10$, $\lambda_2 = 5$

orthonormal eigenbasis $\vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

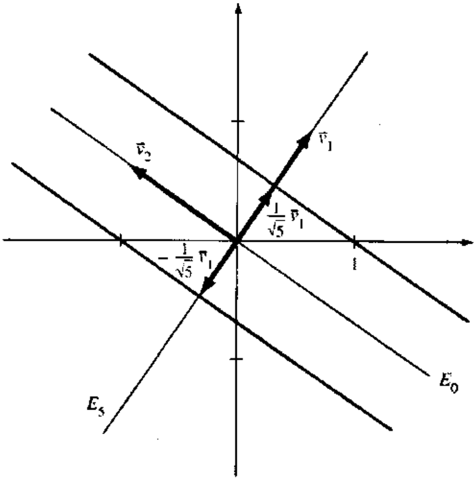
$10c_1^2 + 5c_2^2 = 1$ (ellipse)



19. $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$; eigenvalues $\lambda_1 = 5$, $\lambda_2 = 0$

eigenvectors $\vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$5c_1^2 = 1$ (a pair of lines)

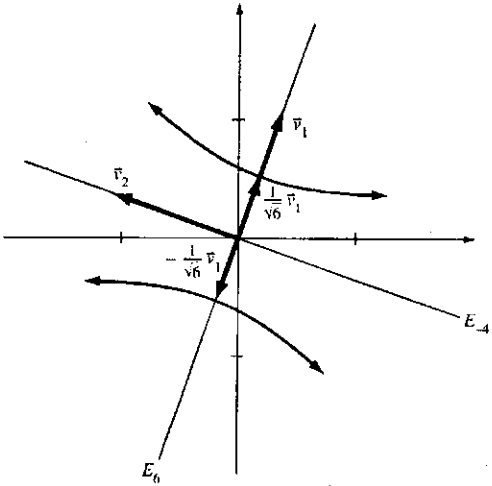


Note that $(x_1^2 + 4x_1x_2 + 4x_2^2) = (x_1 + 2x_2)^2 = 1$, so that $x_1 + 2x_2 = \pm 1$, and the two lines are $x_2 = \frac{1 - x_1}{2}$ and $x_2 = \frac{-1 - x_1}{2}$.

20. $A = \begin{bmatrix} -3 & 3 \\ 3 & 5 \end{bmatrix}$; eigenvalues $\lambda_1 = 6$ and $\lambda_2 = -4$

orthonormal eigenbasis $\vec{v}_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\vec{v}_2 = \frac{1}{\sqrt{10}}\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

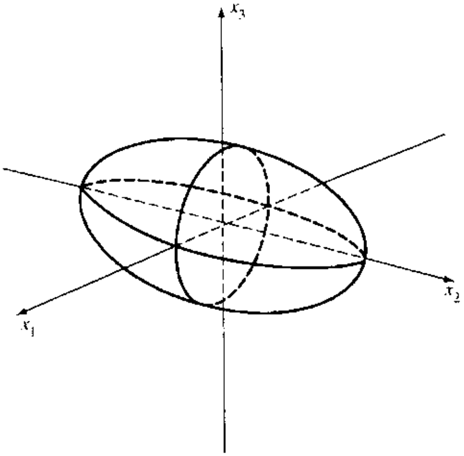
$6c_1^2 - 4c_2^2 = 1$ (hyperbola)



21. a. In each case, it is informative to think about the intersections with the three coordinate planes: $x_1 - x_2$, $x_1 - x_3$, and $x_2 - x_3$.

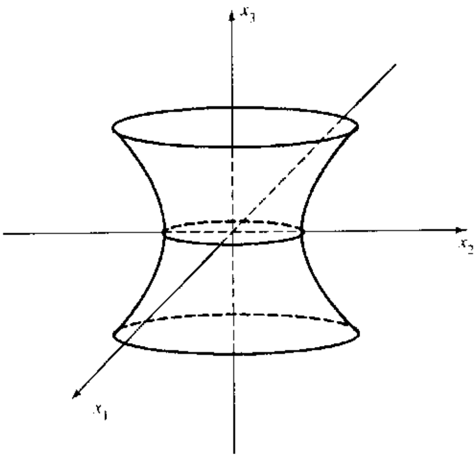
For the surface $x_1^2 + 4x_2^2 + 9x_3^2 = 1$, all these intersections are *ellipses*, and the surface itself is an *ellipsoid*.

This surface is connected and bounded; the points closest to the origin are $\pm \begin{bmatrix} 0 \\ 0 \\ \frac{1}{3} \end{bmatrix}$, and those farthest $\pm \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.



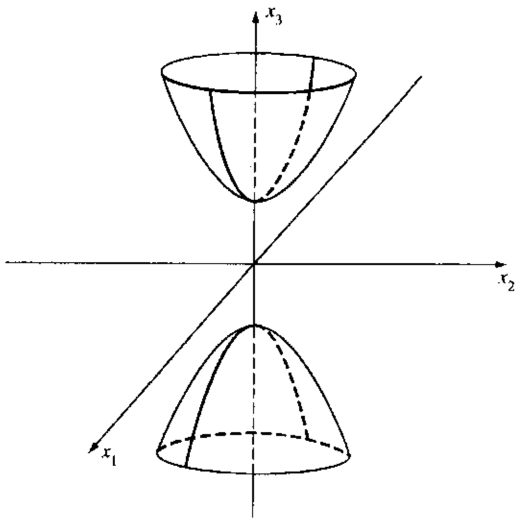
$x_1^2 + 4x_2^2 + 9x_3^2 = 1$ (not to scale)
an *ellipsoid*

In the case of $x_1^2 + 4x_2^2 - 9x_3^2 = 1$, the intersection with the $x_1 - x_2$ plane is an ellipse, and the two other intersections are hyperbolas. The surface is connected and not bounded; the points closest to the origin are $\pm \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}$.



$x_1^2 + 4x_2^2 - 9x_3^2 = 1$ (not to scale)
a *hyperboloid of one sheet*

In the case $-x_1^2 - 4x_2^2 + 9x_3^2 = 1$, the intersection with the $x_1 - x_2$ plane is empty, and the two other intersections are hyperbolas. The surface consists of two pieces and is unbounded. The points closest to the origin are $\pm \begin{bmatrix} 0 \\ 0 \\ \frac{1}{3} \end{bmatrix}$.



$-x_1^2 - 4x_2^2 + 9x_3^2 = 1$ (not to scale)
a *hyperboloid of two sheets*

b. $A = \begin{bmatrix} 1 & \frac{1}{2} & 1 \\ \frac{1}{2} & 2 & \frac{3}{2} \\ 1 & \frac{3}{2} & 3 \end{bmatrix}$ is positive definite, with three positive eigenvalues $\lambda_1, \lambda_2, \lambda_3$.

Surface is given by $\lambda_1 c_1^2 + \lambda_2 c_2^2 + \lambda_3 c_3^2 = 1$ with respect to principal axes, an *ellipsoid*. To find points closest to and farthest from origin, use technology to find eigenvalues and eigenvectors:
eigenvalues: $\lambda_1 \approx 0.56, \lambda_2 \approx 4.44, \lambda_3 = 1$

unit eigenvectors: $\vec{v}_1 \approx \begin{bmatrix} 0.86 \\ 0.19 \\ -0.47 \end{bmatrix}, \vec{v}_2 \approx \begin{bmatrix} 0.31 \\ 0.54 \\ 0.78 \end{bmatrix}, \vec{v}_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

Equation: $0.56c_1^2 + 4.44c_2^2 + c_3^2 = 1$

Farthest points when $c_1 = \pm \frac{1}{\sqrt{0.56}}$ and $c_2 = c_3 = 0$

Closest points when $c_2 = \pm \frac{1}{\sqrt{4.44}}$ and $c_1 = c_3 = 0$

Farthest points $\approx \pm \frac{1}{\sqrt{0.56}} \begin{bmatrix} 0.86 \\ 0.19 \\ -0.47 \end{bmatrix} \approx \pm \begin{bmatrix} 1.15 \\ 0.26 \\ -0.63 \end{bmatrix}$

Closest points $\approx \pm \frac{1}{\sqrt{4.44}} \begin{bmatrix} 0.31 \\ 0.54 \\ 0.78 \end{bmatrix} \approx \pm \begin{bmatrix} 0.15 \\ 0.26 \\ 0.37 \end{bmatrix}$

22. $A = \begin{bmatrix} -1 & 0 & 5 \\ 0 & 1 & 0 \\ 5 & 0 & -1 \end{bmatrix}$; eigenvalues $\lambda_1 = 4$, $\lambda_2 = -6$, $\lambda_3 = 1$

Equation with respect to principal axes: $4c_1^2 - 6c_2^2 + c_3^2 = 1$, a hyperboloid of one sheet (see solutions to 21a).

Closest to origin when $c_1 = \pm\frac{1}{2}$, $c_2 = c_3 = 0$.

A unit eigenvector for eigenvalue 4 is $\vec{v} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, so that the desired points are $\pm\frac{1}{2}\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \approx$

$\pm \begin{bmatrix} 0.35 \\ 0 \\ 0.35 \end{bmatrix}$.