22. We are told that $\frac{d\vec{x}_1}{dt} = A\vec{x}_1$ and $\frac{d\vec{x}_2}{dt} = A\vec{x}_2$. Let $\vec{x}(t) = \vec{x}_1(t) + \vec{x}_2(t)$. Then $\frac{d\vec{x}}{dt} = \frac{d\vec{x}_1}{dt} + \frac{d\vec{x}_2}{dt} = A\vec{x}_1 + A\vec{x}_2 = A(\vec{x}_1 + \vec{x}_2) = A\vec{x}$, as claimed.

23. We are told that $\frac{d\vec{x}_1}{dt} = A\vec{x}_1$. Let $\vec{x}(t) = k\vec{x}_1(t)$. Then $\frac{d\vec{x}}{dt} = \frac{d}{dt}(k\vec{x}_1) = k\frac{d\vec{x}_1}{dt} = kA\vec{x}_1 = A(k\vec{x}_1) = A\vec{x}$, as claimed.

24. We are told that $\frac{d\vec{x}}{dt} = A\vec{x}$. Let $\vec{c}(t) = e^{kt}\vec{x}(t)$. Then $\frac{d\vec{c}}{dt} = \frac{d}{dt}(e^{kt}\vec{x}) = \left(\frac{d}{dt}e^{kt}\right)\vec{x} + e^{kt}\frac{d\vec{x}}{dt} = ke^{kt}\vec{x} + e^{kt}A\vec{x} = (A + kI_n)(e^{kt}\vec{x}) = (A + kI_n)\vec{c}$, as claimed.

25. We are told that $\frac{d\vec{x}}{dt} = A\vec{x}$. Let $\vec{c}(t) = \vec{x}(kt)$. Using the chain rule we find that $\frac{d\vec{c}}{dt} = \frac{d}{dt}(\vec{x}(kt)) = k\left.\frac{d\vec{x}}{dt}\right|_{kt} = kA(\vec{x}(kt)) = kA\vec{c}(t)$, as claimed.

To get the vector field $kA\vec{c}$ we scale the vectors of the field $A\vec{x}$ by k.

26. $\lambda_1 = 3$, $\lambda_2 = -2$; $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, $c_1 = 5, c_2 = -1$, so that $\vec{x}(t) = 5e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - e^{-2t}\begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

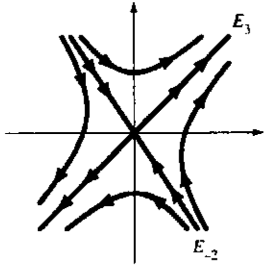
27. Use Fact 9.1.2.

The eigenvalues of $A = \begin{bmatrix} -4 & 3 \\ 2 & -3 \end{bmatrix}$ are $\lambda_1 = -6$ and $\lambda_2 = -1$, with associated eigenvectors $\vec{v}_1 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The coordinates of $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ with respect to $\vec{v}_1$ and $\vec{v}_2$ are $c_1 = -\frac{1}{5}$ and $c_2 = \frac{2}{5}$.

By Fact 9.1.2 the solution is $\vec{x}(t) = -\frac{1}{5}e^{-6t}\begin{bmatrix} -3 \\ 2 \end{bmatrix} + \frac{2}{5}e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

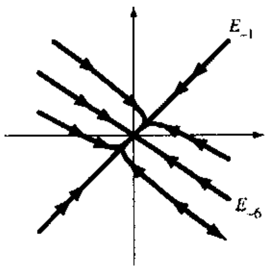
28. $\lambda_1 = 2$, $\lambda_2 = 10$; $\vec{v}_1 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$; $c_1 = -\frac{1}{8}$, $c_2 = \frac{5}{8}$, so that $\vec{x}(t) = -\frac{1}{8}e^{2t}\begin{bmatrix} -3 \\ 2 \end{bmatrix} + \frac{5}{8}e^{10t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

32.



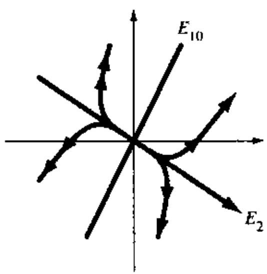
See Exercise 26.

33.



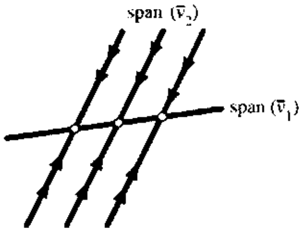
See Exercise 27.

34.



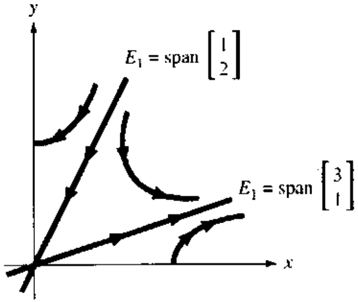
See Exercise 28.

41. The trajectories are of the form $\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 = c_1\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$.



42. a. The term $0.8x$ in the second equation indicates that species y is helped by x, while species x is hindered by y (consider the term $-1.2y$ in the first equation). Thus y preys on x.

b.

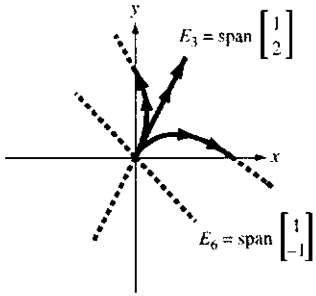


c. If $\dfrac{y(0)}{x(0)} < 2$ then both species will prosper, and $\lim\limits_{t\to\infty} \dfrac{y(t)}{x(t)} = \dfrac{1}{3}$.

If $\dfrac{y(0)}{x(0)} \geq 2$ then both species will die out.

43. a. These two species are *competing* as each is hindered by the other (consider the terms $-y$ and $-2x$).

b.

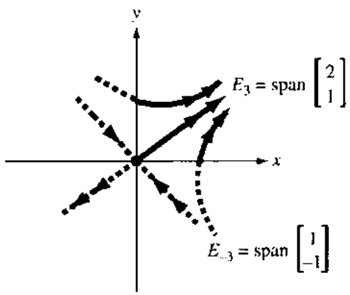


Although only the first quadrant is relevant for our model, it is useful to consider the phase portrait in the other quadrants as well.

c. If $\frac{y(0)}{x(0)} > 2$ then species y wins (x will die out); if $\frac{y(0)}{x(0)} < 2$ then x wins. If $\frac{y(0)}{x(0)} = 2$ then both will prosper and $\frac{y(t)}{x(t)} = 2$ for all t.

44. a. The two species are in symbiosis: Each is helped by the other (consider the terms $4y$ and $2x$).

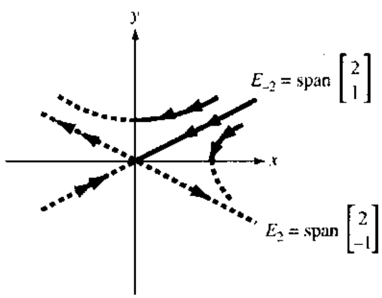
b.



c. Both populations will prosper and $\lim_{t\to\infty} \frac{y(t)}{x(t)} = \frac{1}{2}$, regardless of the initial populations.

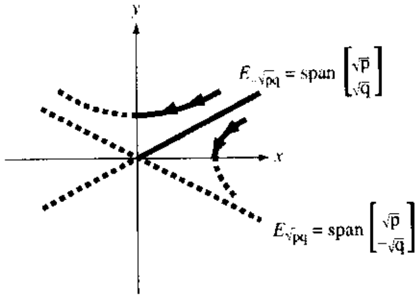
45. a. Species y has the more vicious fighters, since they kill members of species x at a rate of 4 per time unit, while the fighters of species x only kill at a rate of 1.

b.



c. If $\frac{y(0)}{x(0)} < \frac{1}{2}$ then x wins; if $\frac{y(0)}{x(0)} > \frac{1}{2}$ then y wins; if $\frac{y(0)}{x(0)} = \frac{1}{2}$ nobody will survive the battle.

46. *Phase portrait:*

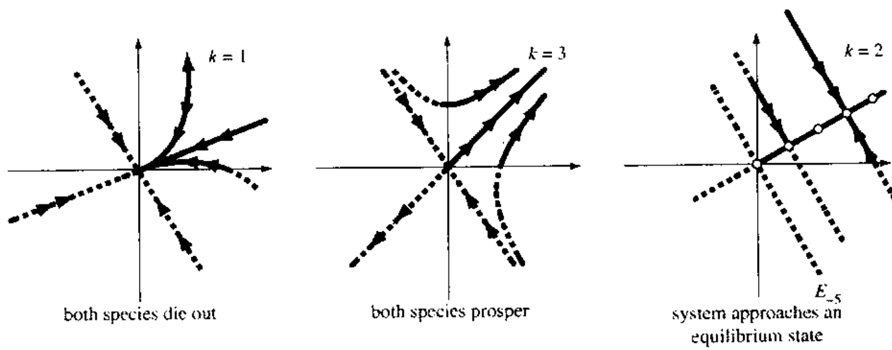


if $\frac{y(0)}{x(0)} < \frac{\sqrt{q}}{\sqrt{p}}$ then x wins; if $\frac{y(0)}{x(0)} > \frac{\sqrt{q}}{\sqrt{p}}$ then y wins; if $\frac{y(0)}{x(0)} = \frac{\sqrt{q}}{\sqrt{p}}$ then nobody will survive the battle.

47. a. The two species are in symbiosis: Each is helped by the other (consider the terms kx and ky).

b. $\lambda_{1,2} = \dfrac{-5 \pm \sqrt{9+4k^2}}{2}$
Both eigenvalues are negative if $\sqrt{9+4k^2} < 5$ or $9 + 4k^2 < 25$ or $4k^2 < 16$ or $k < 2$ (recall that k is positive).
If $k = 2$ then the eigenvalues are -5 and 0.
If $k > 2$ then there is a positive and a negative eigenvalue.

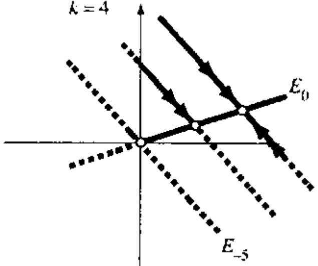
c.



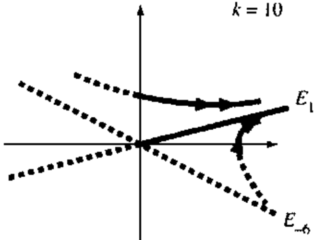
48. a. Symbiosis

b. $\lambda_{1,2} = \dfrac{-5 \pm \sqrt{9+4k}}{2}$
Both eigenvalues are negative if $\sqrt{9+4k} < 5$ or $9 + 4k < 25$ or $4k < 16$ or $k < 4$.
If $k = 4$ then the eigenvalues are -5 and 0.
If $k > 4$ then there is a positive and a negative eigenvalue.

c. $k = 1$: See corresponding figure in Exercise 47.



system approaches
an equilibrium state



system approaches
an equilibrium state

52. The solutions of $\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x}$ are of the form $\begin{bmatrix} p + qt \\ q \end{bmatrix}$, where $\vec{x}(0) = \begin{bmatrix} p \\ q \end{bmatrix}$, by Exercise 21. Since $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = \lambda I_2 + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, the solutions of the given system are of the form $\vec{x}(t) = e^{\lambda t} \begin{bmatrix} p + qt \\ q \end{bmatrix}$, by Exercise 24. The zero state is a stable equilibrium solution if and only if $\lambda < 0$. The case $\lambda = 0$ is discussed in Exercise 21.