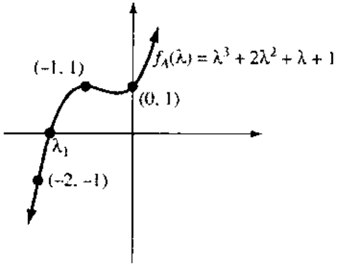
6. $\lambda_{1,2} = \pm i$; $E_i = \ker \begin{bmatrix} i-3 & 2 \\ -5 & i+3 \end{bmatrix} = \text{span} \begin{bmatrix} 3+i \\ 5 \end{bmatrix}$ and $E_{-i} = \text{span} \begin{bmatrix} 3-i \\ 5 \end{bmatrix}$.

General solution:

$\vec{x}(t) = c_1 e^{it} \begin{bmatrix} 3+i \\ 5 \end{bmatrix} + c_2 e^{-it} \begin{bmatrix} 3-i \\ 5 \end{bmatrix}$

If $c_1 = c_2 = 1$ then $\vec{x}(t) = (\cos(t)+i\sin(t)) \begin{bmatrix} 3+i \\ 5 \end{bmatrix} + (\cos(t) - i\sin(t)) \begin{bmatrix} 3-i \\ 5 \end{bmatrix} = \begin{bmatrix} 6\cos(t) - 2\sin(t) \\ 10\cos(t) \end{bmatrix}$.

12. Yes, the zero state is stable. To see this, use technology to determine that the real parts of the eigenvalues are all negative. Or draw a rough sketch of the characteristic polynomial to see that there is one real eigenvalue λ_1 between -1 and -2; the two other eigenvalues must be $\lambda_{2,3} = p \pm iq$ with $p < 0$, since $\lambda_1 + \lambda_2 + \lambda_3 = \lambda_1 + 2p = \operatorname{tr}(A) = -2$.



22. $\lambda_1 = 3, \lambda_2 = 0.5; E_3 = \text{span}\begin{bmatrix} 1 \\ -1 \end{bmatrix}, E_{0.5} = \text{span}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

System is discrete so choose VII.

23. $\lambda_{1,2} = -\frac{1}{2} \pm i, r > 1$, so that trajectory spirals outwards. Choose II.

24. $\lambda_1 = 3, \lambda_2 = 0.5, E_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, E_{0.5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

System is continuous, so choose I.

25. $\lambda_{1,2} = -\frac{1}{2} \pm i$; real part is negative so that trajectories spiral inwards in the counterclockwise direction $\left(\text{if } \vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ then } \frac{d\vec{x}}{dt} = \begin{bmatrix} -1.5 \\ 2 \end{bmatrix}\right)$. Choose IV.

26. $\lambda_1 = 1, \lambda_2 = -2; E_1 = \text{span}\begin{bmatrix} 0 \\ 1 \end{bmatrix}, E_{-2} = \text{span}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

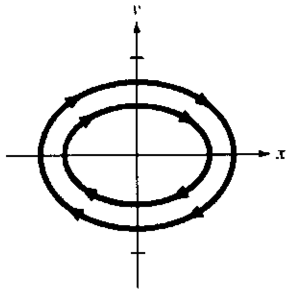
System is continuous so choose V.

30. $\lambda_{1,2} = -2 \pm 3i$, $E_{-2+3i} = \text{span}\left(\begin{bmatrix} 5 \\ 3 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$, so that

$$\vec{x}(t) = e^{-2t} \begin{bmatrix} 0 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \cos(3t) & -\sin(3t) \\ \sin(3t) & \cos(3t) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = e^{-2t} \begin{bmatrix} 5\sin(3t) & 5\cos(3t) \\ \cos(3t) + 3\sin(3t) & -\sin(3t) + 3\cos(3t) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

35. $A = \begin{bmatrix} 0 & 1 \\ -b & -c \end{bmatrix}$ and $f_A(\lambda) = \lambda^2 + c\lambda + b$, with eigenvalues $\lambda_{1,2} = \dfrac{-c \pm \sqrt{c^2 - 4b}}{2}$.

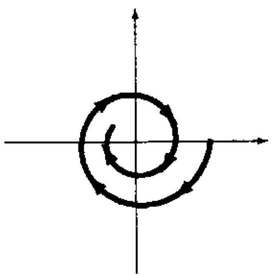
a. If $c = 0$ then $\lambda_{1,2} = \pm i\sqrt{b}$. The trajectories are ellipses.



The block *oscillates harmonically*, with period $\dfrac{2\pi}{\sqrt{b}}$. The zero state is not asymptotically stable.

b. $\lambda_{1,2} = \dfrac{-c \pm i\sqrt{4b - c^2}}{2}$

The trajectories spiral inwards, since $\text{Re}(\lambda_1) = \text{Re}(\lambda_2) = -\dfrac{c}{2} < 0$. This is the case of a *damped oscillation*. The zero state is asymptotically stable.



c. This case is discussed in Exercise 9.1.55. The zero state is stable here.

39. Let $A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$. We first solve the system $\frac{d\vec{c}}{dt} = (A - \lambda I_3)\vec{c} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \vec{c}$, or $\frac{dc_1}{dt} = c_2(t)$, $\frac{dc_2}{dt} = c_3(t)$, $\frac{dc_3}{dt} = 0$.

$c_3(t) = k_3$, a constant, so that $\frac{dc_2}{dt} = k_3$ and $c_2(t) = k_3 t + k_2$. Likewise $c_1(t) = \frac{k_3}{2}t^2 + k_2 t + k_1$.

Applying Exercise 9.1.24, with $k = -\lambda$, we find that $\vec{c}(t) = e^{-\lambda t}\vec{x}(t)$ or $\vec{x}(t) = e^{\lambda t}\vec{c}(t)$

$= e^{\lambda t} \begin{bmatrix} k_1 + k_2 t + \frac{k_3}{2}t^2 \\ k_2 + k_3 t \\ k_3 \end{bmatrix}$ where k_1, k_2, k_3 are arbitrary constants. The zero state is stable if (and only if) the real part of λ is negative.