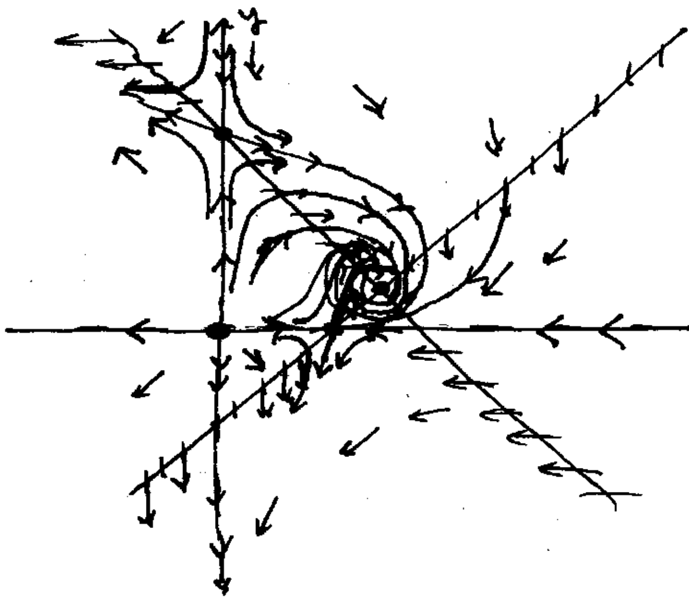
SECTION 9.4 SOLUTIONS. SUPPLEMENT ON NONLINEAR SYSTEMS

① $\frac{dx}{dt} = x(2-x+y) = 2x - x^2 + xy = f(x,y)$ $\quad$ $\frac{dx}{dt} = 0$ when $\underline{\underline{x=0}}$ or $\underline{\underline{y = x-2}}$

$\frac{dy}{dt} = y(4-x-y) = 4y - xy - y^2 = g(x,y)$ $\quad$ $\frac{dy}{dt} = 0$ when $\underline{\underline{y=0}}$ or $\underline{\underline{y = -x+4}}$



$x \geq 0$
$y \geq 0$

EQUILIBRIA AT
$(0,0)$ $\quad$ $(0,4)$
$(2,0)$ $\quad$ $(3,1)$

$$J_F = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} 2-2x+y & x \\ -y & 4-x-2y \end{bmatrix}$$

$$J_F(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

evals $\lambda_1 = 2$, $\lambda_2 = 4$
$\Rightarrow$ source, unstable

$$J_F(0,4) = \begin{bmatrix} 6 & 0 \\ -4 & -4 \end{bmatrix} \qquad \begin{bmatrix} \lambda-6 & 0 \\ 4 & \lambda+4 \end{bmatrix}$$

$(\lambda-6)(\lambda+4) = 0$

$\lambda_1 = 6$ $\quad$ $\lambda_2 = -4$ evalues.

evectors $\vec{v}_1 = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$ $\quad$ $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$J_F(2,0) = \begin{bmatrix} -2 & 2 \\ 0 & 2 \end{bmatrix}$$

$\begin{bmatrix} \lambda+2 & -2 \\ 0 & \lambda-2 \end{bmatrix}$ $\quad$ $(\lambda+2)(\lambda-2) = 0$

$\lambda_1 = 2$ $\quad$ $\lambda_2 = -2$

evectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\quad$ $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$J_F(3,1) = \begin{bmatrix} -3 & 3 \\ -1 & -1 \end{bmatrix} \qquad \begin{bmatrix} \lambda+3 & -3 \\ 1 & \lambda+1 \end{bmatrix}$$

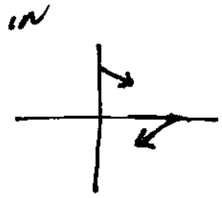
$\lambda^2 + 4\lambda + 3 + 3 = \lambda^2 + 4\lambda + 6 = 0$

$$\lambda = \frac{-4 \pm \sqrt{16-24}}{2}$$

$\lambda = -2 + \sqrt{2}\,i$ $\quad$ $\bar{\lambda} = -2 - \sqrt{2}\,i$

$Re(\lambda) < 0 \Rightarrow$ SPIRAL IN

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} -3 \\ -1 \end{bmatrix}$ $\quad$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 3 \\ -1 \end{bmatrix}$



$\Rightarrow$ clockwise.

② $\frac{dx}{dt} = x(1-x+ky-k) = x - x^2 + kxy - kx \qquad k \neq 1 \quad k \neq -1$

$\frac{dy}{dt} = y(1-y+kx-k) = y - y^2 + kxy - ky$

EQUILIBRIA where $x=0$ OR $1-x+ky-k=0$

AND

$y=0$ OR $1-y+kx-k=0$

$\Rightarrow$ Ⓐ $(0,0)$ Ⓑ $x=0$ and $1-y+kx-k=0$ $\Rightarrow y = 1-k \Rightarrow (0, 1-k)$

Ⓒ $y=0$ and $1-x+ky-k=0$ $\Rightarrow x = 1-k$ $\Rightarrow (1-k, 0)$

Ⓓ $\begin{matrix} -x+ky = k-1 \\ kx - y = k-1 \end{matrix} \Rightarrow \begin{bmatrix} -1 & k \\ k & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k-1 \\ k-1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{1-k^2}\begin{bmatrix} -1 & -k \\ -k & -1 \end{bmatrix}\begin{bmatrix} k-1 \\ k-1 \end{bmatrix} = \frac{1}{1-k^2}\begin{bmatrix} -k+1-k^2+k \\ -k^2+k-k+1 \end{bmatrix}$$

$$= \frac{1}{1-k^2}\begin{bmatrix} 1-k^2 \\ 1-k^2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\Rightarrow \boxed{(1,1)}$ in 1st Quadrant

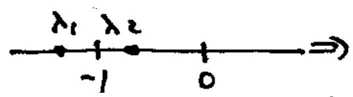
$$J_F = \begin{bmatrix} 1-2x+ky-k & kx \\ ky & 1-2y+kx-k \end{bmatrix} \qquad J_F(1,1) = \begin{bmatrix} -1 & k \\ k & -1 \end{bmatrix}$$

$$\begin{bmatrix} \lambda+1 & -k \\ -k & \lambda+1 \end{bmatrix} \qquad \lambda^2 + 2\lambda + 1 - k^2 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{4 - 4 + 4k^2}}{2}$$

$$\Rightarrow \boxed{\lambda = -1 \pm |k|}$$

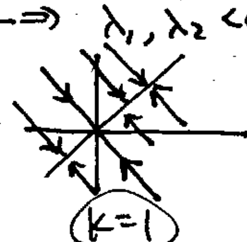
If $k=0 \Rightarrow \lambda = -1, -1$ STABLE

If $|k| < 1 \Rightarrow$  $\Rightarrow \lambda_1, \lambda_2 < 0 \Rightarrow$ STABLE

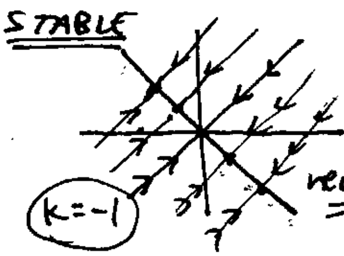
If $|k| = 1 \Rightarrow \lambda_1 = -2, \lambda_2 = 0$

$k=1 \Rightarrow \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$k=-1 \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$



OR



neutral

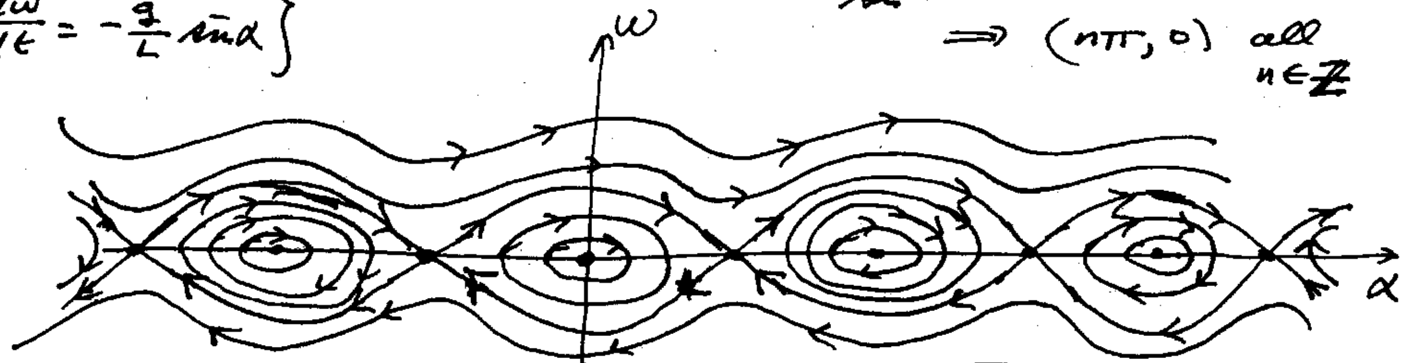
If $|k| > 1 \quad \lambda_1 < 0, \lambda_2 > 0 \Longrightarrow$  unstable

③ $\begin{cases} \dfrac{d\alpha}{dt} = \omega \\ \dfrac{d\omega}{dt} = -\dfrac{g}{L}\sin\alpha \end{cases}$

EQUILIBRIA AT $\omega = 0$

$\sin\alpha = 0 \Rightarrow \alpha = n\pi$

$\Rightarrow (n\pi, 0)$ all $n \in \mathbb{Z}$



$J_P = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\cos\alpha & 0 \end{bmatrix}$

n even $\Rightarrow J_F(n\pi, 0) = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix}$

$\begin{bmatrix} \lambda & -1 \\ \frac{g}{L} & \lambda \end{bmatrix}$ $\quad \lambda^2 + \frac{g}{L} = 0$

$\lambda = \pm i\sqrt{\frac{g}{L}}$

complex, $Re(\lambda) = 0$.

LOCAL PICTURE

n odd $\Rightarrow J_F(n\pi, 0) = \begin{bmatrix} 0 & 1 \\ g/L & 0 \end{bmatrix}$ $\quad \begin{bmatrix} \lambda & -1 \\ -g/L & \lambda \end{bmatrix}$

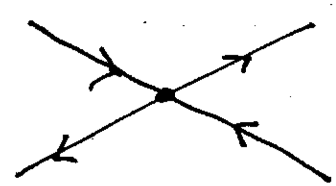
$\lambda^2 - \frac{g}{L} = 0 \qquad \lambda = \pm\sqrt{g/L} \qquad \lambda_1 > 0$, $\lambda_2 < 0$

unstable.

$\lambda_1 = +\sqrt{g/L} \Rightarrow \begin{bmatrix} \sqrt{g/L} & -1 \\ -g/L & \sqrt{g/L} \end{bmatrix} \longrightarrow \sqrt{g/L}\,\alpha = \beta \Rightarrow \vec{V}_1 = \begin{bmatrix} 1 \\ \sqrt{g/L} \end{bmatrix}$

$\lambda_2 = -\sqrt{g/L} \Rightarrow \vec{V}_2 = \begin{bmatrix} 1 \\ -\sqrt{g/L} \end{bmatrix}$.

LOCAL PICTURE :



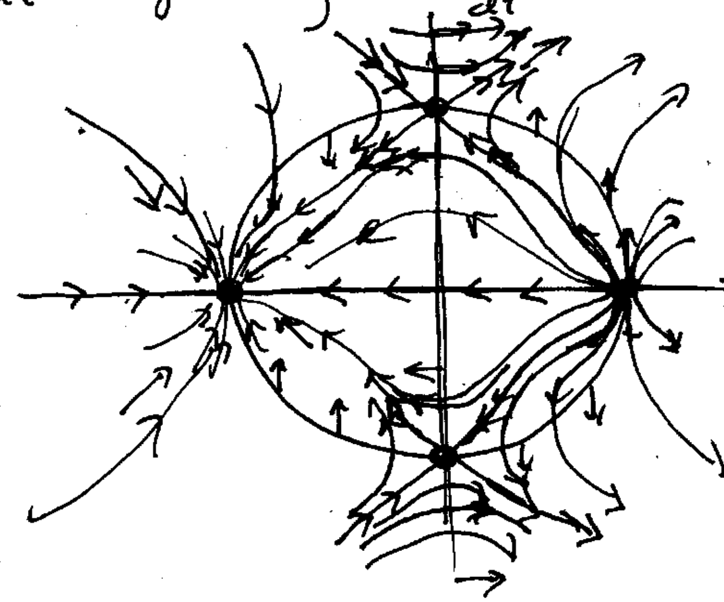
In phase diagram at top,
upper curves represent pendulum goin round and round.

CLOSED TRAJECTORIES represent OSCILLATIONS.
Lower curves repr. pendulum goin round and round the other way.

④ $\begin{cases} \dfrac{dx}{dt} = x^2+y^2-1 \\ \dfrac{dy}{dt} = xy \end{cases}$

$\frac{dx}{dt} = 0 \Rightarrow$ CIRCLE $x^2+y^2=1$

$\frac{dy}{dt} = 0 \Rightarrow x=0$ OR $y=0$



PHASE PORTRAIT

EQUILIBRIA AT $(0,1)$ $(0,-1)$ $(1,0)$ $(-1,0)$

$$J_F = \begin{bmatrix} 2x & 2y \\ y & x \end{bmatrix}$$

$$J_F(0,1) = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$

$\begin{bmatrix} \lambda & -2 \\ -1 & \lambda \end{bmatrix}$ $\lambda^2-2=0$

$\lambda_1=\sqrt{2}$ $\lambda_2=-\sqrt{2}$

(saddle)

$$J_F(0,-1) = \begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix}$$

$\begin{bmatrix} \lambda & 2 \\ 1 & \lambda \end{bmatrix}$ $\lambda^2-2=0$

$\lambda_1=\sqrt{2}$ $\lambda_2=-\sqrt{2}$

(SADDLE)

$$J_F(1,0) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$\lambda_1 = 2$ $\lambda_2 = 1$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(source)

unstable

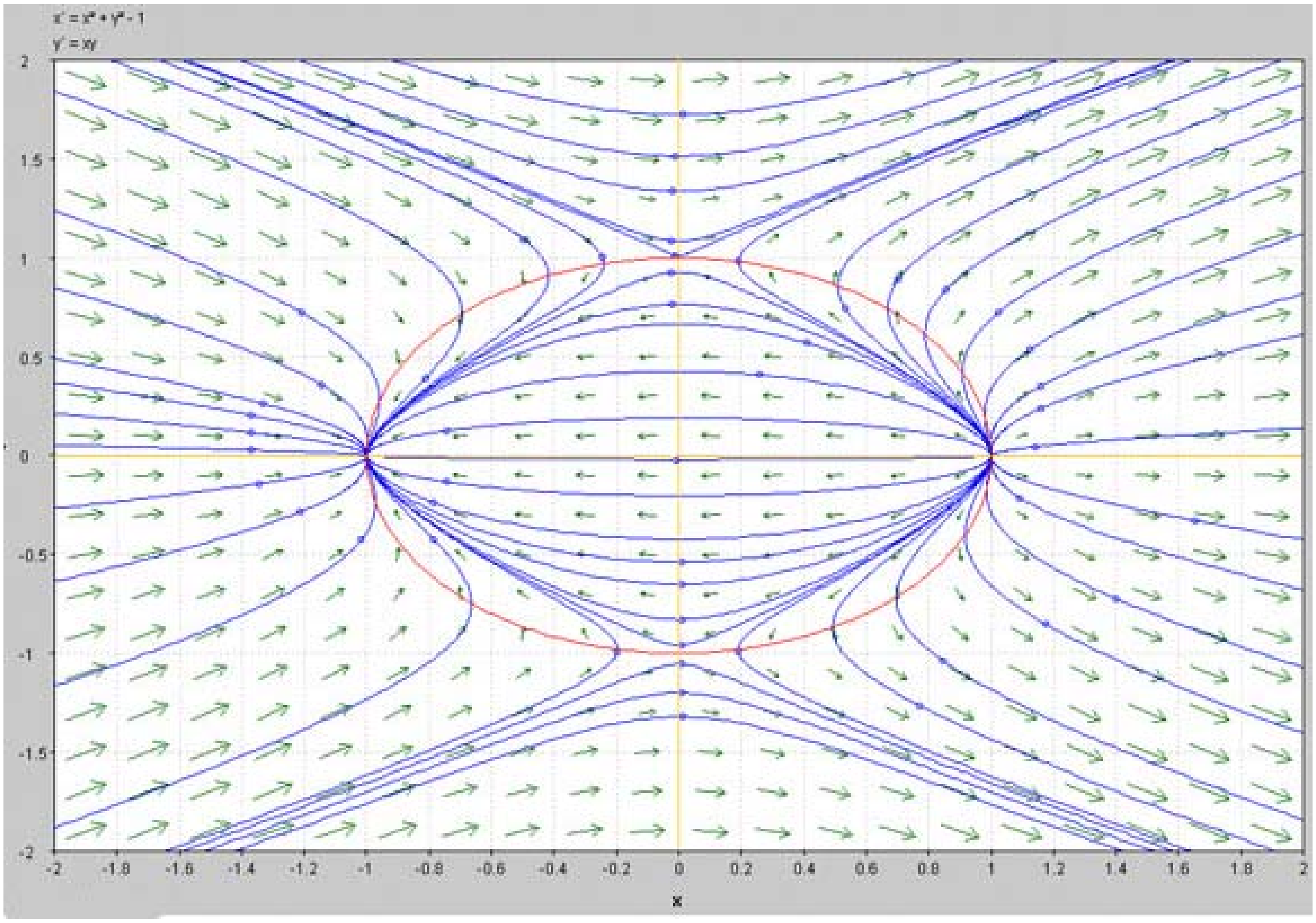
$$J_F(-1,0) = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$$

$\lambda_1 = -2$ $\lambda_2 = -1$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

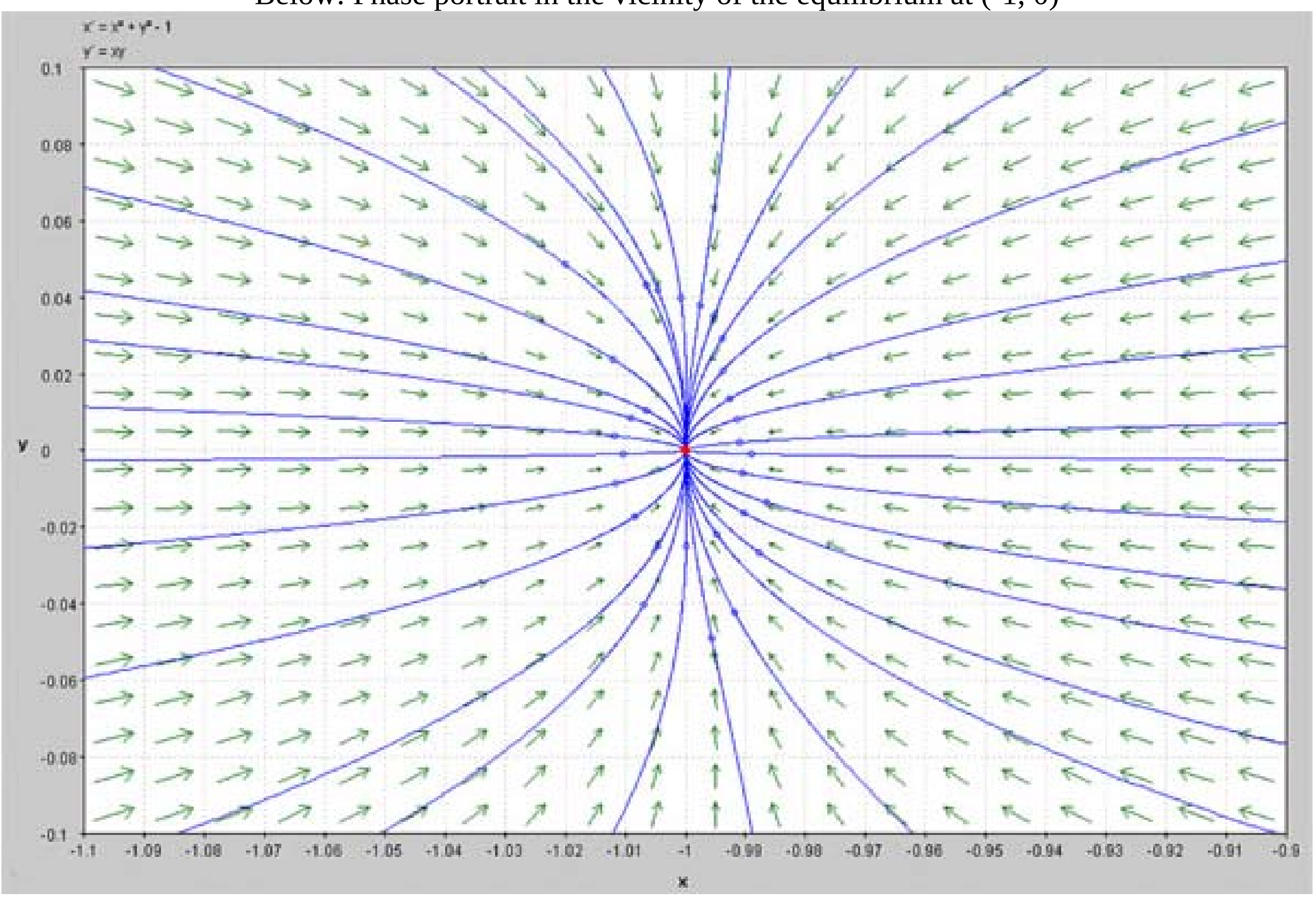
(SINK)

STABLE



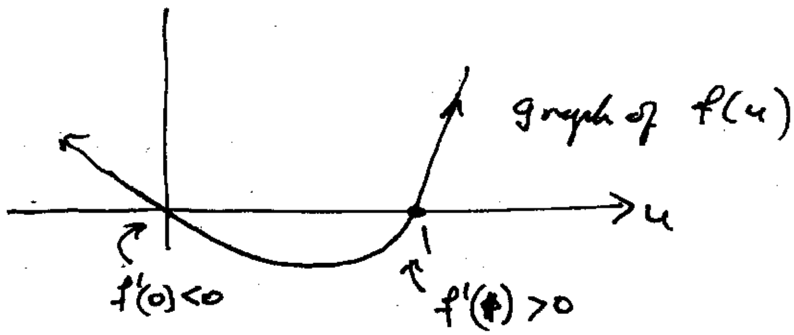
Above: Phase portrait for problem 4

Below: Phase portrait in the vicinity of the equilibrium at (-1, 0)



⑤ $\frac{d^2u}{dt^2} = f(u) - \frac{du}{dt}$ Let $v = \frac{du}{dt}$

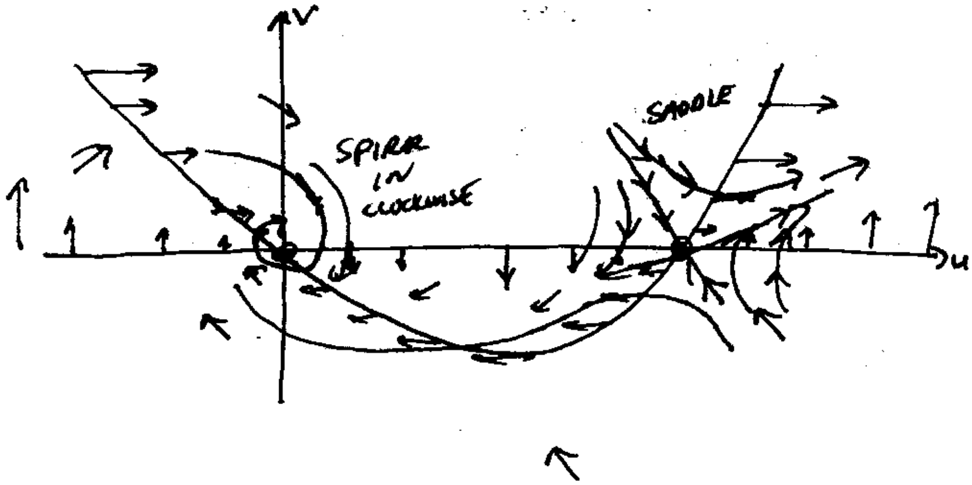
$$\Rightarrow \begin{cases} \frac{du}{dt} = v \\ \frac{dv}{dt} = f(u) - v \end{cases}$$



NULLCLINES:

$\frac{du}{dt} = 0$ when $v = 0$

$\frac{dv}{dt} = 0$ when $f(u) - v = 0 \Rightarrow v = f(u)$



EQUILIBRIA AT $(0,0)$ and $(1,0)$

$$J_F = \begin{bmatrix} 0 & 1 \\ f'(u) & -1 \end{bmatrix}$$

$$J_F(0,0) = \begin{bmatrix} 0 & 1 \\ - & -1 \end{bmatrix}$$

trace $= \lambda_1 + \lambda_2 = -1$

det $= \lambda_1 \lambda_2 = +$ $\Rightarrow$ λ_1, λ_2 both positive OR both negative.

If λ complex $\lambda = a + ib$

$\lambda \bar{\lambda} = a^2 + b^2 > 0$ ✓

$\lambda_1 + \lambda_2 = 2a \Rightarrow Re(\lambda) = -\frac{1}{2}$

$\lambda_1 + \lambda_2 = -1 \Rightarrow$ both negative (if Real)

$\Rightarrow$ <u>STABLE</u>

$$J_F(1,0) = \begin{bmatrix} 0 & 1 \\ + & -1 \end{bmatrix}$$

trace $= -1 = \lambda_1 + \lambda_2$

det $= \lambda_1 \lambda_2 = -$ $\quad \lambda_1 \lambda_2 < 0$

$\Rightarrow \lambda_1 > 0 \quad \lambda_2 < 0$

$\Rightarrow$ SADDLE $\Rightarrow$ <u>unstable</u>