SOLVE SYSTEM:

$$\left\{\begin{aligned}
\frac{dx_1}{dt} &= 2x_1 - 4x_4 + 3x_5 \\
\frac{dx_2}{dt} &= 2x_2 - 2x_3 + 2x_4 \\
\frac{dx_3}{dt} &= x_2 - x_4 \\
\frac{dx_4}{dt} &= -x_4 \\
\frac{dx_5}{dt} &= -3x_4 + 2x_5
\end{aligned}\right\}$$

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

$$A = \begin{bmatrix} 2 & 0 & 0 & -4 & 3 \\ 0 & 2 & -2 & 2 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 & 2 \end{bmatrix}$$

Eigenvalues:

$$\lambda I - A = \begin{bmatrix} \lambda-2 & 0 & 0 & 4 & -3 \\ 0 & \lambda-2 & 2 & -2 & 0 \\ 0 & -1 & \lambda & 1 & 0 \\ 0 & 0 & 0 & \lambda+1 & 0 \\ 0 & 0 & 0 & 3 & \lambda-2 \end{bmatrix}$$

use Laplace expansion for determinant using rows with all but one zero.

$$\det(\lambda I - A) = (\lambda-2)\left[(\lambda-2)(\lambda+1)(\lambda^2-2\lambda+2)\right]$$

$$= (\lambda-2)^2(\lambda+1)(\lambda^2-2\lambda+2) = 0$$

evals are $\lambda_1 = \lambda_2 = 2$ $\quad \lambda_3 = -1$ $\quad \lambda_{4,5} = \dfrac{2 \pm \sqrt{4-8}}{2}$

$\Rightarrow \lambda_4 = 1 + i \quad \lambda_5 = \overline{\lambda_4} = 1 - i$

$$\lambda_1 = \lambda_2 = 2 \Rightarrow \left[\begin{array}{ccccc|c} 0 & 0 & 0 & 4 & -3 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & -1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \cancel{x_2 = x_3 = x_4 = x_5 = 0}$$

$$\Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Geom. mult $= 1 < 2 =$ Alg. MULT.

CANNOT FIND ANOTHER INDEP. E-VECTOR, But we ~~can~~ find a 2nd vector $\vec{v}_2$ such that $A\vec{v}_2 = \vec{v}_1 + \lambda\vec{v}_2$.

$\Rightarrow \lambda\vec{v}_2 - A\vec{v}_2 = -\vec{v}_1 \Rightarrow (\lambda I - A)\vec{v}_2 = -\vec{v}_1$. (with $\lambda = 2$).

$$\Rightarrow \left[\begin{array}{ccccc|c} 0 & 0 & 0 & 4 & -3 & -1 \\ 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & -1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} t \\ 0 \\ 0 \\ 0 \\ 1/3 \end{bmatrix}$$

For simplicity, choose $t = 0$

$$\Rightarrow \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1/3 \end{bmatrix}$$

$$\lambda_3 = -1 \Rightarrow \left[\begin{array}{ccccc|c} -3 & 0 & 0 & 4 & -3 & 0 \\ 0 & -3 & 2 & -2 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & -3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1/3 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

pick $x_5 = 3$

$$\Rightarrow \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

$\Longrightarrow$

$$\lambda_4 = 1+i \Rightarrow \left[\begin{array}{ccccc|c} -1+i & 0 & 0 & 4 & -3 & 0 \\ 0 & -1+i & 2 & -2 & 0 & 0 \\ 0 & -1 & 1+i & 1 & 0 & 0 \\ 0 & 0 & 0 & 2+i & 0 & 0 \\ 0 & 0 & 0 & 3 & -1+i & 0 \end{array}\right] \to \left[\begin{array}{ccccc|c} -1+i & 0 & 0 & 4 & -3 & 0 \\ 0 & -1+i & 2 & -2 & 0 & 0 \\ 0 & -1 & 1+i & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & -1+i & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccccc|c} -1+i & 0 & 0 & 0 & -3 & 0 \\ 0 & -1+i & 2 & 0 & 0 & 0 \\ 0 & -1 & 1+i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1+i & 2 & 0 & 0 & 0 \\ 0 & -1 & 1+i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

If $\vec{v} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \\ \varepsilon \end{bmatrix}$ is complex eigenvector

then $\alpha = 0,\ \delta = 0,\ \varepsilon = 0$ $\quad (-1+i)\beta + 2\gamma = 0$
or $\quad -\beta + (1+i)\gamma = 0$ (easier)

$\Rightarrow \beta = (1+i)\gamma$ $\quad$ pick $\gamma = 1 \Rightarrow \beta = 1+i$ $\qquad \vec{v} = \begin{bmatrix} 0 \\ 1+i \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$$\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \vec{x} + i\vec{y}.$$

choose $\vec{v}_4 = \vec{y}$
$\vec{v}_5 = \vec{x}$.

CHANGE OF BASIS MATRIX is $S = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 1/3 & 3 & 0 & 0 \end{bmatrix}$

This gives $S^{-1}AS = [A]_{\mathcal{B}} = \left[\begin{array}{cc|c|cc} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] = B$

$\mathcal{B} = \{\vec{v}_1, \dots \vec{v}_5\}$

$$\Rightarrow \boxed{\begin{array}{l} \left\{\begin{array}{l} \dfrac{du_1}{dt} = 2u_1 + u_2 \\ \dfrac{du_2}{dt} = 2u_2 \end{array}\right\} \\ \dfrac{du_3}{dt} = -u_3 \\ \left\{\begin{array}{l} \dfrac{du_4}{dt} = u_4 - u_5 \\ \dfrac{du_5}{dt} = u_4 + u_5 \end{array}\right\} \end{array}}$$

we've solved each of these cases individually

$\vec{u}(t) = \left[e^{tB}\right] \vec{u}(0)$ where

$$e^{tB} = \left[\begin{array}{cc|c|cc} e^{2t} & te^{2t} & 0 & 0 & 0 \\ 0 & e^{2t} & 0 & 0 & 0 \\ \hline 0 & 0 & e^{-t} & 0 & 0 \\ \hline 0 & 0 & 0 & e^{t}\cos t & -e^{t}\sin t \\ 0 & 0 & 0 & e^{t}\sin t & e^{t}\cos t \end{array}\right]$$

FINALLY, $A = SBS^{-1} \Rightarrow \left[e^{tA}\right] = S\left[e^{tB}\right]S^{-1}$

so $\vec{x}(t) = \left[e^{tA}\right]\vec{x}(0)$

$$= S\, e^{tB}\, \underbrace{S^{-1}\vec{x}(0)}_{\begin{bmatrix} c_1 \\ \vdots \\ c_5 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 1/3 & 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{2t} & te^{2t} & 0 & 0 & 0 \\ 0 & e^{2t} & 0 & 0 & 0 \\ 0 & 0 & e^{-t} & 0 & 0 \\ 0 & 0 & 0 & e^t\cos t & -e^t\sin t \\ 0 & 0 & 0 & e^t\sin t & e^t\cos t \end{bmatrix} \begin{array}{c} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{array}$$

$$= \begin{bmatrix} e^{2t} & te^{2t} & e^{-t} & 0 & 0 \\ 0 & 0 & 0 & e^t\cos t + e^t\sin t & -e^t\sin t + e^t\cos t \\ 0 & 0 & 3e^{-t} & e^t\sin t & e^t\cos t \\ 0 & 0 & 3e^{-t} & 0 & 0 \\ 0 & \frac{1}{3}e^{2t} & 3e^{-t} & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix}$$

$$\Rightarrow \left\{ \begin{aligned} x_1(t) &= c_1 e^{2t} + c_2 t e^{2t} + c_3 e^{-t} \\ x_2(t) &= c_4(e^t\cos t + e^t\sin t) + c_5(-e^t\sin t + e^t\cos t) \\ x_3(t) &= 3c_3 e^{-t} + c_4 e^t \sin t + c_5 e^t \cos t \\ x_4(t) &= 3c_3 e^{-t} \\ x_5(t) &= \tfrac{1}{3}c_2 e^{2t} + 3c_3 e^{-t} \end{aligned} \right\} \text{General Solution (part a)}$$

In particular, if $\vec{x}(0) = \begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$, then $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = S^{-1}\vec{x}(0)$

$$= \begin{bmatrix} 1 & 0 & 0 & -1/3 & 0 \\ 0 & 0 & 0 & -3 & 3 \\ 0 & 0 & 0 & 1/3 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 13/3 \\ -3 \\ 2/3 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix}$$

$$\Rightarrow \boxed{\begin{aligned} x_1(t) &= \tfrac{13}{3}e^{2t} - 3te^{2t} + \tfrac{2}{3}e^{-t} \\ x_2(t) &= e^t(4\cos t + 2\sin t) \\ x_3(t) &= 2e^{-t} + e^t(3\sin t + \cos t) \\ x_4(t) &= 2e^{-t} \\ x_5(t) &= -e^{2t} + 2e^{-t} \end{aligned}}$$

(part b)