

True or False

1. T, by Fact 5.3.4.a
2. F. We have $(AB)^T = B^T A^T$, by Fact 5.3.9.a
3. T, since $(A + B)^T = A^T + B^T = A + B$
4. T, by Fact 5.3.4
5. F. Consider $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.
6. T. First note that $A^T = A^{-1}$, by Fact 2.4.9. Thus A is orthogonal, by Fact 5.3.7
7. F. The correct formula is $\text{proj}_L(\vec{x}) = (\vec{x} \cdot \vec{v})\vec{v}$, by Fact 2.2.5.
8. T, since $(7A)^T = 7A^T = 7A$.
9. F. Consider $T(\vec{x}) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \vec{x}$.
10. T, by Fact 5.3.9.b
11. T, since $(ABBA)^T = A^T B^T B^T A^T = ABBA$, by Fact 5.3.9.a
12. T, since $A^T B^T = (BA)^T = (AB)^T = B^T A^T$, by Fact 5.3.9.a
13. F. $\dim(V) + \dim(V^\perp) = 5$, by Fact 5.4.2.a. Thus one of the dimensions is even and the other odd.
14. T. Consider the QR factorization (Fact 5.2.2)
15. F. The Pythagorean Theorem holds for orthogonal vectors \vec{x}, \vec{y} only (Fact 5.1.8)
16. T, by Definition 5.3.1. Note that $1 = \left\| A \left(\frac{1}{\|\vec{x}\|} \vec{x} \right) \right\| = \left\| \frac{1}{\|\vec{x}\|} A\vec{x} \right\| = \frac{1}{\|\vec{x}\|} \|A\vec{x}\|$ for all nonzero \vec{x} , so that $\|A\vec{x}\| = \|\vec{x}\|$.
17. T. If A is orthogonal, then $A^T A = I_n$, by Fact 5.3.7, so that $(A^T)^T = A = (A^T)^{-1}$. Again by Fact 5.3.7, this implies that matrix A^T is orthogonal.
18. F. Consider $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.
19. T, since $\vec{x} \cdot (\text{proj}_V \vec{x}) = (\text{proj}_V \vec{x} + (\vec{x} - \text{proj}_V \vec{x})) \cdot \text{proj}_V \vec{x} = \|\text{proj}_V \vec{x}\|^2 \geq 0$. Note that $\vec{x} - \text{proj}_V \vec{x}$ is orthogonal to $\text{proj}_V \vec{x}$, by the definition of a projection.
20. F. It is required that the columns of A be orthonormal (Fact 5.3.10). As a counterexample, consider $A = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, with $AA^T = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$.

21. T, since the columns are unit vectors.
22. T. Use the Gram-Schmidt process to construct such a basis (Algorithm 5.2.1)
23. F. The columns fail to be unit vectors (use Fact 5.3.3.b)
24. T, by definition of an orthogonal projection (Fact 5.1.6).
25. F. As a counterexample, consider $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.
26. T, by Fact 5.4.1.
27. T, by Fact 5.4.3.a.
28. F. Consider $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, or any other symmetric matrix that fails to be orthogonal.
29. T. If $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is a symmetric matrix, then $A - xI_2 = \begin{bmatrix} a-x & b \\ b & c-x \end{bmatrix}$. This matrix fails to be invertible if (and only if) $\det(A - xI_2) = (a-x)(c-x) - b^2 = 0$. We use the quadratic formula to find the (real) solutions $x = \frac{a+c \pm \sqrt{(a+c)^2 - 4ac + 4b^2}}{2} = \frac{a+c \pm \sqrt{(a-c)^2 + 4b^2}}{2}$. Note that the discriminant $(a-c)^2 + 4b^2$ is positive or zero.
30. F. If A fails to be square, then $\frac{1}{2}(A - A^T)$ is undefined. The claim does hold for square matrices, however, since $[\frac{1}{2}(A - A^T)]^T = \frac{1}{2}(A - A^T)^T = \frac{1}{2}(A^T - A) = -[\frac{1}{2}(A - A^T)]$.
31. F. Consider $A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$.
32. T. By Fact 5.1.11, quantity $\cos(\alpha) = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \|\vec{w}\|}$ is positive, so that α is an acute angle.
33. T. In Fact 5.4.1, let $A = B^T$ to see that $(\text{im}(B^T))^\perp = \ker(B)$. Now take the orthogonal complements of both sides and use Fact 5.4.2.b.
34. T, since $(A^T A)^T = A^T (A^T)^T = A^T A$, by Fact 5.3.9.a.
35. F. Verify that matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ are similar.
36. F. Consider $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. The correct formula $\text{im}(B) = \text{im}(BB^T)$ follows from Facts 5.4.1 and 5.4.3.
37. T. We know that $A^T = A$ and $S^{-1} = S^T$. Now $(S^{-1}AS)^T = S^T A^T (S^{-1})^T = S^{-1}AS$, by Fact 5.3.9.a.

38. T. By Fact 5.4.3, we have $\ker(A) = \ker(A^T A)$. Replacing A by A^T in this formula, we find that $\ker(A^T) = \ker(AA^T)$. Now $\ker(A) = \ker(A^T A) = \ker(AA^T) = \ker(A^T)$.
39. T. Try $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$, so that $A + B = \begin{bmatrix} 1 + \cos \alpha & -\sin \alpha \\ \sin \alpha & 1 + \cos \alpha \end{bmatrix}$. It is required that $\begin{bmatrix} 1 + \cos \alpha \\ \sin \alpha \end{bmatrix}$ and $\begin{bmatrix} -\sin \alpha \\ 1 + \cos \alpha \end{bmatrix}$ be unit vectors, meaning that $1 + 2 \cos \alpha + \cos^2 \alpha + \sin^2 \alpha = 2 + 2 \cos \alpha = 1$, or $\cos \alpha = -\frac{1}{2}$, and $\sin \alpha = \pm \frac{\sqrt{3}}{2}$. Thus $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$ is a solution.
40. F. Consider $A = \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$, for example, representing a rotation-dilation.
41. T. We attempt to write $A = S + Q$, where S is symmetric and Q is skew-symmetric. Then $A^T = S^T + Q^T = S - Q$. Adding the equations $A = S + Q$ and $A^T = S - Q$ together gives $2S = A + A^T$ and $S = \frac{1}{2}(A + A^T)$. Similarly we find $Q = \frac{1}{2}(A - A^T)$. Check that the decomposition $A = S + Q = (\frac{1}{2}(A + A^T)) + (\frac{1}{2}(A - A^T))$ does the job.
42. T. Apply the Cauchy-Schwarz inequality (squared), $(\vec{x} \cdot \vec{y})^2 \leq \|\vec{x}\|^2 \|\vec{y}\|^2$, to $\vec{x} = \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 1 \\ \dots \\ 1 \end{bmatrix}$ (all n entries are 1).
43. T. Let $A = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$. We know that $AA^T = A^2$, or $\begin{bmatrix} x^2 + y^2 & xz + yt \\ xz + yt & z^2 + t^2 \end{bmatrix} = \begin{bmatrix} x^2 + yz & xy + yt \\ zx + tz & yz + t^2 \end{bmatrix}$. We need to show that $y = z$. If $y \neq 0$, this follows from the equation $x^2 + y^2 = x^2 + yz$; if $z \neq 0$, it follows from $z^2 + t^2 = yz + t^2$; if both y and z are zero, we are all set.