True or False

1. T, by Fact 6.1.3 (a diagonal matrix is triangular as well)

2. T, by Fact 6.2.4

3. T; See work on Page 243

4. F; We have $\det(4A) = 4^4 \det(A)$, by Fact 6.2.4

5. F; Let $A = B = I_5$, for example

6. T; We have $\det(-A) = (-1)^6 \det(A) = \det(A)$, by Fact 6.2.4

7. F; If fact, $\det(A) = 0$, since A is noninvertible

8. F; The matrix A fails to be invertible if $\det(A) = 0$, by Fact 6.2.5

9. T, by Fact 6.2.4

10. T, by Fact 6.2.7

11. T, by Example 6 of Section 6.2

12. F, by Fact 6.3.1. The determinant can be -1.

13. T, by Fact 6.2.7.

14. F: The second and the fourth column are linearly dependent.

15. F; If $k = -1$ or $k = -2$, then the matrix has two equal columns, so that it fails to be invertible.

16. F; The determinant is -1, since there are three inversions.

17. T; The pattern with the four entries 100 guarantees a nonzero determinant.

18. F; Let $A = 2I_2$, for example

19. T; Matrix A is invertible

20. T; Any nonzero noninvertible matrix A will do.

21. T, by Fact 6.3.4.

22. T; We have $\det(A) = \det(\text{rref } A) = 0$.

23. F; Let $A = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$, for example

24. F; Let $A = 2I_2$, for example

25. T; Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$. The columns of A are orthogonal; now use Fact 6.3.4.

26. F; Let $A = \begin{bmatrix} 8 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, for example.

27. F; In fact, $\vec{v} \cdot (\vec{u} \times \vec{w}) = -(\det A)$.

28. T; Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, for example.

29. F; Note that $\det \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = 2$.

30. T, by Fact 6.3.10

31. F; Let $A = 2I_2$, for example

32. F; Let $A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, for example.

33. F; Let $A = I_2$ and $B = -I_2$, for example.

34. T; Note that $\det(B) = -\det(A) < \det(A)$, so that $\det(A) > 0$.

35. T; Let's do Laplace expansion along the first row, for example (see Fact 6.2.10).

Then $\det(A) = \sum_{j=1}^{n}(-1)^{1+j}a_{1j}\det(A_{1j}) \neq 0$. Thus $\det(A_{1j}) \neq 0$ for at least one j, so that A_{1j} is invertible.

36. T; Note that $\det(A)$ and $\det(A^{-1})$ are both integers, and $(\det A)(\det A^{-1}) = 1$. This leaves only the possibilities $\det(A) = \det(A^{-1}) = 1$ and $\det(A) = \det(A^{-1}) = -1$.

37. T, since $\operatorname{adj}(A) = (\det A)(A^{-1})$, by Fact 6.3.10.

38. F; Note that $\det(A^2) = (\det A)^2$ cannot be negative, but $\det(-I_3) = -1$.

39. F; Note that $\det(S^{-1}AS) = \det(A)$ but $\det(2A) = 2^3(\det A) = 8(\det A)$.

40. F; Note that $\det(S^TAS) = (\det S)^2(\det A)$ and $\det(-A) = -(\det A)$ have opposite signs.

41. T; The diagonal pattern makes an odd contribution to the determinant, while all the other contributions are even. Thus the determinant is odd; in particular, it is nonzero.

42. F; Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 5 & 2 \end{bmatrix}$, for example

43. T; Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $a \neq 0$, let $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$; if $b \neq 0$, let $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$; if $c \neq 0$, let $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and if $d \neq 0$, let $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

44. T; Use Gaussian elimination for the first column only to transform A into a matrix of the form

$$B = \begin{bmatrix} 1 & \pm 1 & \pm 1 & \pm 1 \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix}$$

Note that $\det(B) = \det(A)$ or $\det(B) = -(\det A)$. The stars in matrix B all represent numbers $(\pm 1) \pm (\pm 1)$, so that they are 2, 0, or -2. Thus the determinant of the 3×3 matrix M containing the stars is divisible by 8, since each pattern makes a contribution of 8, 0, or -8. Now perform Laplace expansion down the first column of B to see that $\det(B) = \det(M)$.