

Math S-21b – Summer 2005 – Exam #1 Solutions

1) True or False

a) If \mathbf{A} is a 5×4 matrix of rank 4, then the kernel of \mathbf{A} must be nonzero.

FALSE – The matrix \mathbf{A} represents a linear transformation from \mathbf{R}^4 to \mathbf{R}^5 . In the case where \mathbf{A} is of maximal rank, i.e., $\text{rank}(\mathbf{A}) = 4$, the rank-nullity theorem says that nullity = 0 (since rank + nullity = 4 for a 5×4 matrix). So the kernel of \mathbf{A} must be $\{\mathbf{0}\}$.

b) Let \mathbf{A} be a 5×4 matrix such that $\dim(\ker \mathbf{A}) = 1$. Then for any \mathbf{b} in \mathbf{R}^5 , the solution set to $\mathbf{Ax} = \mathbf{b}$ is a line in \mathbf{R}^4 parallel to $\ker \mathbf{A}$.

FALSE – Since \mathbf{A} is a 5 by 4 matrix, it represents a linear transformation from \mathbf{R}^4 to \mathbf{R}^5 . Since we are given that $\dim(\ker \mathbf{A}) = 1$, we can apply the Rank-Nullity Theorem to conclude that $\dim(\text{im } \mathbf{A}) = 3$. It is therefore quite possible that given any \mathbf{b} in \mathbf{R}^5 , it will not lie in the image. Consequently, the system would be inconsistent and there would be no solutions.

c) Let \mathbf{A} be an invertible 3×3 matrix and let $V = \{\mathbf{v} : \|\mathbf{Av}\| \leq 1\}$, that is the set of all vectors whose images under \mathbf{A} have length less than or equal to 1. V is a subspace of \mathbf{R}^3 .

FALSE – Such a subset cannot possibly be a subspace because it is not closed under scalar multiplication. For example, if \mathbf{v} is such that the magnitude of \mathbf{Av} is 1, then $\mathbf{A}(2\mathbf{v}) = 2\mathbf{Av}$ will have a magnitude of 2, so $2\mathbf{v}$ will not be in the subset V .

d) Consider the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ in \mathbf{R}^m . Let \mathbf{A} be a $p \times m$ matrix. If the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly dependent, then so are the vectors $\mathbf{Av}_1, \mathbf{Av}_2, \dots, \mathbf{Av}_n$.

TRUE – If vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ are linearly dependent, then there are constants c_1, \dots, c_n , not all zero, such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n = \mathbf{0}$. Apply (multiply by) the matrix \mathbf{A} on both sides to get:

$\mathbf{A}(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n) = c_1\mathbf{Av}_1 + c_2\mathbf{Av}_2 + \dots + c_n\mathbf{Av}_n = \mathbf{0}$. Thus, there's a nontrivial linear combination of the vectors $\{\mathbf{Av}_1, \mathbf{Av}_2, \dots, \mathbf{Av}_n\}$ that sums to the zero vector, so they are also a linearly dependent set of vectors.

e) $\text{rank}(\mathbf{A}^2) \leq \text{rank}(\mathbf{A})$ for any square matrix \mathbf{A} . (\mathbf{A}^2 denotes the product \mathbf{AA} .)

TRUE – Use the fact that $\text{im}(\mathbf{A}^2) \subseteq \text{im}(\mathbf{A})$ to conclude that $\dim(\text{im}(\mathbf{A}^2)) \leq \dim(\text{im}(\mathbf{A}))$.

2) Let \mathbf{A} be the 4×5 matrix $\mathbf{A} = \begin{bmatrix} 1 & -2 & 1 & 5 & 0 \\ 1 & -2 & -1 & 1 & 0 \\ 3 & -6 & -1 & 7 & -1 \\ -1 & 2 & 0 & -3 & 0 \end{bmatrix}$.

a) Find all solutions of the equation $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 3 \\ 5 \\ 1 \\ -4 \end{bmatrix}$.

Solution: This is solved using row reduction on the augmented matrix:

$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & 5 & 0 & 3 \\ 1 & -2 & -1 & 1 & 0 & 5 \\ 3 & -6 & -1 & 7 & -1 & 1 \\ -1 & 2 & 0 & -3 & 0 & -4 \end{array} \right] \xrightarrow{RREF} \left[\begin{array}{ccccc|c} 1 & -2 & 0 & 3 & 0 & 4 \\ 0 & 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 12 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

There are “leading 1’s in the 1st, 3rd, and 5th columns of the reduced augmented matrix, so we’ll be able to solve for the corresponding variables and introduce parameters for the others. Thus, we can choose $x_2 = s$ and $x_4 = t$ and solve for the others. So all solutions are of the form:

$$\left\{ \begin{array}{l} x_1 = 4 + 2s - 3t \\ x_2 = s \\ x_3 = -1 - 2t \\ x_4 = t \\ x_5 = 12 \end{array} \right\} \text{ or in vector form } \mathbf{x} = \begin{bmatrix} 4 \\ 0 \\ -1 \\ 0 \\ 12 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}.$$

The solution to this inhomogeneous system is just a translate of $\ker(\mathbf{A})$, the solution to the homogeneous system.

b) Find a basis for the kernel of \mathbf{A} and its dimension, i.e. the nullity of \mathbf{A} .

Solution: We solve the homogeneous system $\mathbf{Ax} = \mathbf{0}$ by row reduction. This is precisely the same calculation as above only with all zeros on the right side:

$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & 5 & 0 & 0 \\ 1 & -2 & -1 & 1 & 0 & 0 \\ 3 & -6 & -1 & 7 & -1 & 0 \\ -1 & 2 & 0 & -3 & 0 & 0 \end{array} \right] \xrightarrow{RREF} \left[\begin{array}{ccccc|c} 1 & -2 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

So all solutions are of the form:

$$\left\{ \begin{array}{l} x_1 = 2s - 3t \\ x_2 = s \\ x_3 = -2t \\ x_4 = t \\ x_5 = 0 \end{array} \right\}, \text{ or in vector form, } \mathbf{x} = s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}.$$

Thus the kernel is of dimension 2 and these two vectors form a basis for $\ker(\mathbf{A})$.

c) Find a basis for the image of \mathbf{A} and its dimension, i.e. the rank of \mathbf{A} .

Solution: We’ve shown that a basis for $\text{im}(\mathbf{A})$ can be chosen by taking only those column vectors of the original matrix \mathbf{A} that led to a “leading 1” in $\text{rref}(\mathbf{A})$. Thus, in this example we may choose the 1st, 3rd, and 5th column vectors of \mathbf{A} , namely:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \end{bmatrix}, \text{ and } \mathbf{v}_5 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}.$$

Thus the dimension of the image, i.e. the rank of \mathbf{A} , must be 3.

3) Suppose the linear transformation $T: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ is clockwise rotation by 90° and that $S: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ is orthogonal (perpendicular) projection onto the line $L = \text{Span} \left\{ \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$. Find the matrix (relative to the standard basis) of the transformation $S \circ T$, that is, the rotation T followed by the projection S .

Solution: Clockwise rotation sends the vector \mathbf{e}_1 to $-\mathbf{e}_2$, and \mathbf{e}_2 to \mathbf{e}_1 , so the matrix of T is $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. There are several good methods for calculating the matrix of the projection, but they all yield that this matrix is $\frac{1}{25} \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$. The matrix of the composition $S \circ T$ is just the product of the corresponding matrices (in the same order), so we calculate:

$$\frac{1}{25} \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \frac{1}{25} \begin{bmatrix} -12 & 9 \\ -16 & 12 \end{bmatrix} = \begin{bmatrix} -.48 & .36 \\ -.64 & .48 \end{bmatrix}$$

4) The vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ are linearly independent (and are therefore a basis for \mathbf{R}^3).

a) If $\mathbf{A}\mathbf{v}_1 = \mathbf{v}_2$, $\mathbf{A}\mathbf{v}_2 = \mathbf{v}_1$, and $\mathbf{A}\mathbf{v}_3 = 2\mathbf{v}_3$ (where $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are given above), determine the matrix \mathbf{A} (relative to the standard basis).

Solution 1: Substituting the given vectors into these three relations, we have:

$$\mathbf{A} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{A} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \text{and} \quad \mathbf{A} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}.$$

These three can be rolled into one matrix statement, namely $\mathbf{A} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 2 \\ 1 & 2 & 2 \end{bmatrix}$.

Calling these two matrices \mathbf{S} and \mathbf{B} , we can write this as $\mathbf{A}\mathbf{S} = \mathbf{B}$. If we further note that \mathbf{S} is invertible (because the column vectors $\mathbf{v}_1, \mathbf{v}_2$, and \mathbf{v}_3 are linearly independent), we can solve for \mathbf{A} as $\mathbf{A} = \mathbf{B}\mathbf{S}^{-1}$. We calculate this inverse matrix using row reduction or a calculator and do the matrix multiplication thus:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 2 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -1 \\ -4 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ -6 & -2 & 4 \\ -3 & 0 & 2 \end{bmatrix}.$$

Solution 2: The given relations allow us to easily write down the matrix of this linear transformation

relation to the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, namely $[\mathbf{A}]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$, where \mathbf{S} is the change-of-basis

matrix $\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$. From this we calculate:

$$\mathbf{A} = \mathbf{S}[\mathbf{A}]_{\mathcal{B}}\mathbf{S}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -1 \\ -4 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 2 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -1 \\ -4 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ -6 & -2 & 4 \\ -3 & 0 & 2 \end{bmatrix}.$$

b) Find a nonzero vector \mathbf{x} such that $\mathbf{Ax} = -\mathbf{x}$. [There are infinitely many such solutions. Think about it!]

Solution 1: If you simply treat this as a system of linear equations, you get:

$$\left\{ \begin{array}{l} 2x_1 + x_2 - x_3 = -x_1 \\ -6x_1 - 2x_2 + 4x_3 = -x_2 \\ -3x_1 + 2x_3 = -x_3 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} 3x_1 + x_2 - x_3 = 0 \\ -6x_1 - x_2 + 4x_3 = 0 \\ -3x_1 + 3x_3 = 0 \end{array} \right\} \Rightarrow \left[\begin{array}{ccc|c} 3 & 1 & -1 & 0 \\ -6 & -1 & 4 & 0 \\ -3 & 0 & 3 & 0 \end{array} \right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

The solutions are therefore of the form: $\begin{cases} x_1 = t \\ x_2 = -2t \\ x_3 = t \end{cases}$, or $\mathbf{x} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. In particular, $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is a solution.

Solution 2: We can do the same solution as above with matrices: $\mathbf{Ax} = -\mathbf{x} = -\mathbf{Ix}$. This gives us $\mathbf{Ax} + \mathbf{Ix} = \mathbf{0}$ or, more simply, $(\mathbf{A} + \mathbf{I})\mathbf{x} = \mathbf{0}$, and this is solved by row reduction as above.

Solution 3: If you note that $\mathbf{Av}_1 = \mathbf{v}_2$ and $\mathbf{Av}_2 = \mathbf{v}_1$, you can observe that:

$$\mathbf{A}(\mathbf{v}_1 - \mathbf{v}_2) = \mathbf{Av}_1 - \mathbf{Av}_2 = \mathbf{v}_2 - \mathbf{v}_1 = -(\mathbf{v}_1 - \mathbf{v}_2).$$

Therefore, the vector $(\mathbf{v}_1 - \mathbf{v}_2)$ is a solution. (It's the same as given above in Solution 1.)

c) Find the coordinates of the vector \mathbf{x} you found in part (b) relative to the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of \mathbf{R}^3 .

Solution: The coordinates of the vector \mathbf{x} relative to the basis \mathcal{B} are found by:

$$[\mathbf{x}]_{\mathcal{B}} = \mathbf{S}^{-1}\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -1 \\ -4 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Note that this just says that $\mathbf{x} = \mathbf{v}_1 - \mathbf{v}_2$.