Math S-21b – Summer 2005 – Exam #2 Solutions

1) TRUE/FALSE

a) The matrix $\begin{bmatrix} \cos\theta & -\sin\theta & 1 \\ \sin\theta & \cos\theta & 2 \\ 0 & 0 & 3 \end{bmatrix}$ is invertible for all θ.

TRUE: The determinant of the matrix is 3. Since this is nonzero, the matrix is invertible.

b) $\det(\mathbf{A}^T\mathbf{A}^2\mathbf{A}^T) = -16$ for some matrix **A**. [Here $\mathbf{A}^T$ denotes the transpose of **A** and the entries of **A** are presumed to be real numbers.]
FALSE: $\det(\mathbf{A}^T\mathbf{A}^2\mathbf{A}^T) = \det(\mathbf{A}^T)\det(\mathbf{A}^2)\det(\mathbf{A}^T) = \det(\mathbf{A})[\det(\mathbf{A})]^2\det(\mathbf{A}) = [\det(\mathbf{A})]^4 = -16$. This is clearly impossible for a matrix **A** (with real entries) since its determinant will be real and there is no real number which, when raised to the 4^{th} power, will give a negative number.

c) If T is a linear transformation from $\mathbf{R}^n$ to $\mathbf{R}^n$ which sends orthogonal vectors to orthogonal vectors, then T is an orthogonal transformation.
FALSE: For example, a dilation that scales by anything other than 1 or –1 will preserve orthogonality but not norm, and will therefore not be an orthogonal transformation.

d) Let **A** be an orthogonal matrix. Then det(**A**) = 1.
FALSE: The determinant of an orthogonal matrix can be either +1 or –1.

e) A projection is an orthogonal transformation.
FALSE: The kernel of a projection is generally a nonzero subspace, so it's not invertible, the determinant is zero, and norm is not preserved.

f) If **A** is a $n \times n$ matrix such that $\mathbf{A}^2 = 2\mathbf{A} - \mathbf{I}$, then the only possible eigenvalue of **A** is $\lambda = 1$.
TRUE: Suppose **v** is an eigenvector of **A** with eigenvalue λ. The given relation can be rewritten as $\mathbf{A}^2 - 2\mathbf{A} + \mathbf{I} = \mathbf{0}$, the zero matrix. Therefore, $(\mathbf{A}^2 - 2\mathbf{A} + \mathbf{I})\mathbf{v} = \mathbf{A}^2\mathbf{v} - 2\mathbf{A}\mathbf{v} + \mathbf{v} = (\lambda^2 - 2\lambda + 1)\mathbf{v} = \mathbf{0}$, so $(\lambda^2 - 2\lambda + 1) = (\lambda - 1)^2 = 0$. Thus the only possible eigenvalue of **A** is 1.

g) If P_2 is the linear space consisting of all polynomials of degree ≤ 2, and $T: P_2 \to P_2$ is defined by $T(f) = f'' - 2f' + 3f$, then T is an isomorphism.
TRUE: There are several ways to show that T is an isomorphism. For example, by looking at what T does to the basis $\{1, t, t^2\}$, we can get its matrix relative to this basis, namely $\begin{bmatrix} 3 & -2 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 3 \end{bmatrix}$. It's determinant is not zero, so the linear transformation is invertible.

h) A real $n \times n$ matrix **A** is called *skew-symmetric* if $\mathbf{A}^T = -\mathbf{A}$. If λ is an eigenvalue of **A**, then –λ is also an eigenvalue of **A**.
TRUE: Suppose **v** is an eigenvector of **A** with eigenvalue λ. Therefore $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$. Since $\mathbf{A}^T = -\mathbf{A}$, it follows that $\det(\lambda\mathbf{I} + \mathbf{A}^T) = 0$. Since the determinant of the matrix $\lambda\mathbf{I} + \mathbf{A}$ and its transpose are the same, we have that $\det(\lambda\mathbf{I} + \mathbf{A}) = 0$. Multiplying this matrix by –1, we also have that $\det(-\lambda\mathbf{I} - \mathbf{A}) = 0$. Therefore –λ must also be an eigenvalue.

2) We are given the 4 points (-3, 3), (–1, 2), (1, 2), and (2, 1).

a) Write down the (inconsistent) system of equations corresponding to all four of these points satisfying the linear equation $y = mx + b$.

b) Find the normal equation (in matrix form) whose solution is the least-squares solution of this inconsistent system of equations; and

c) Solve the normal equation and give the values for the slope m and the intercept b for this best-fit line.

SOLUTION: If we substitute each of these four points into the equation $y = mx + b$, we get the four inconsistent linear equations $\left\{\begin{array}{r} -3m+b=3 \\ -m+b=2 \\ m+b=2 \\ 2m+b=1 \end{array}\right\}$. In matrix form, this reads $\mathbf{Ac} = \begin{bmatrix} -3 & 1 \\ -1 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 2 \\ 1 \end{bmatrix} = \mathbf{y}$.

The least-squares solution is found by solving the <u>normal equation</u> $\mathbf{A}^{\mathrm{T}}\mathbf{Ac} = \mathbf{A}^{\mathrm{T}}\mathbf{y}$.

This gives us $\begin{bmatrix} 15 & -1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} -7 \\ 8 \end{bmatrix}$, which we solve to get $m = -{}^{20}\!/_{59} \cong -.339$ and $b = {}^{113}\!/_{59} \cong 1.915$.

3) We are given two vectors in $\mathbf{R}^4$, $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ -2 \\ 1 \\ 6 \end{bmatrix}$.

a) Find the area of the parallelogram determined by the vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$.

SOLUTION: We use the fact that Area = $\sqrt{\det(\mathbf{A}^{\mathrm{T}}\mathbf{A})}$ where the columns of $\mathbf{A}$ are the given vectors.

$$\mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & 2 \\ 2 & -2 & 1 & 6 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -1 & -2 \\ 0 & 1 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 9 & 18 \\ 18 & 45 \end{bmatrix}$$. Therefore the area is $\sqrt{81} = 9$.

b) Find an <u>orthonormal</u> basis $\{\mathbf{w}_1, \mathbf{w}_2\}$ for the subspace Span$\{\mathbf{v}_1, \mathbf{v}_2\}$, using the Gram-Schmidt orthogonalization method.

SOLUTION: $\|\mathbf{v}_1\| = 3$, so $\mathbf{w}_1 = \dfrac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \dfrac{1}{3}\begin{bmatrix} 2 \\ -1 \\ 0 \\ 2 \end{bmatrix}$.

If we subtract the projection of $\mathbf{v}_2$ in the direction, we get

$$\mathbf{v}_2 - (\mathbf{v}_2 \bullet \mathbf{w}_1)\mathbf{w}_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \\ 6 \end{bmatrix} - \tfrac{1}{3}(4+2+0+12)\tfrac{1}{3}\begin{bmatrix} 2 \\ -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \\ 6 \end{bmatrix} - \begin{bmatrix} 4 \\ -2 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 2 \end{bmatrix}.$$

Normalizing this, we get $\mathbf{w}_2 = \dfrac{1}{3}\begin{bmatrix} -2 \\ 0 \\ 1 \\ 2 \end{bmatrix}$. So the orthonormal basis is $\left\{\dfrac{1}{3}\begin{bmatrix} 2 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \dfrac{1}{3}\begin{bmatrix} -2 \\ 0 \\ 1 \\ 2 \end{bmatrix}\right\}$.

c) Find the matrix for orthogonal projection onto the subspace spanned by the two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

SOLUTION: Using the <u>orthonormal</u> basis for V found in part (b), we let $\mathbf{B} = \begin{bmatrix} \uparrow & \uparrow \\ \mathbf{w}_1 & \mathbf{w}_2 \\ \downarrow & \downarrow \end{bmatrix} = \dfrac{1}{3}\begin{bmatrix} 2 & -2 \\ -1 & 0 \\ 0 & 1 \\ 2 & 2 \end{bmatrix}$.

We've previously shown that the matrix for orthogonal projection onto V is:

$$\mathbf{P} = \mathbf{BB}^{\mathrm{T}} = \frac{1}{9}\begin{bmatrix} 2 & -2 \\ -1 & 0 \\ 0 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 & 2 \\ -2 & 0 & 1 & 2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 8 & -2 & -2 & 0 \\ -2 & 1 & 0 & -2 \\ -2 & 0 & 1 & 2 \\ 0 & -2 & 2 & 8 \end{bmatrix}.$$

4) a) Find the eigenvalues and eigenvectors of the matrix $\mathbf{A} = \begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}$.

SOLUTION: $\lambda\mathbf{I} - \mathbf{A} = \begin{bmatrix} \lambda-8 & 10 \\ -5 & \lambda+7 \end{bmatrix}$, so the characteristic polynomial is

$$p_{\mathbf{A}}(\lambda) = \lambda^2 - \lambda - 6 = (\lambda-3)(\lambda+2) = 0 .$$

The eigenvalues are therefore 3 and –2. A quick calculation gives the respective eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and the change of basis matrix is $\mathbf{S} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and its inverse $\mathbf{S}^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$. The corresponding diagonal matrix is $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{D} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$.

b) Give a closed form expression for $\mathbf{A}^t\mathbf{x}_0$ where $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and t is any (positive integer) power.

SOLUTION: We have that $\mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$ and $\mathbf{A}^t = \mathbf{S}\mathbf{D}^t\mathbf{S}^{-1}$, so we calculate:

$$\mathbf{A}^t\mathbf{x}_0 = \mathbf{S}\mathbf{D}^t\mathbf{S}^{-1}\mathbf{x}_0 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3^t & 0 \\ 0 & (-2)^t \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2(3^t) & (-2)^t \\ 3^t & (-2)^t \end{bmatrix}\begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} -4(3^t) + 5(-2)^t \\ -2(3^t) + 5(-2)^t \end{bmatrix} = -2 \cdot 3^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 5 \cdot (-2)^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$