Math S-21b – Summer 2005

Exam Topics and Practice Exam for 2nd Midterm Exam

Exam Topics

1) Basic ideas of general linear spaces: basis, dimension, matrix of a linear transformation relative to a basis; examples of spaces of matrices, spaces of functions, etc. (sections 4.1-4.3)

2) Orthogonality (perpendicularity) of vectors in R^n; length (norm) of a vector, unit vectors; orthogonal complements; orthogonal projections; orthonormal basis; angle between two vectors; Gram-Schmidt orthogonalization process; QR factorization; orthogonal transformation; orthogonal matrix. (sections 5.1-5.3)

3) Least-squares approximation; normal equation. (section 5.4)

4) Determinant of a (square) matrix, patterns and permutations; Laplace expansion; multilinearity and the effect of the row operations on the value of the determinant; determinant criterion for invertibility of a matrix; k-volumes; determinant as an expansion factor. (sections 6.1-6.3)

5) Discrete (linear) dynamical systems, iteration of a matrix, trajectories and phase portraits; eigenvectors and eigenvalues of a (square) matrix; characteristic polynomial; algebraic and geometric multiplicities; diagonalization and the existence of a basis of eigenvectors; powers of a matrix; trace and determinant. (sections 7.1-7.4)

Note that Inner Product Spaces and Fourier Series will not be covered on this exam.

Some specifics:

1) Describe what it means for two $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ to be *similar*.

2) Given a general (finite dimensional) linear space V (not just $\mathbf{R}^n$) described in words, find a basis for V.

3) Given an element $\mathbf{x} \in V$, a general linear space, find the coordinates of $\mathbf{x}$ relative to a given basis for V.

4) Given a linear transformation T: $V \to V$, and a basis $\mathbb{B}$ for V, find the matrix of T relative to the basis $\mathbb{B}$.

5) Given a subspace $V \subseteq \mathbf{R}^n$, find its orthogonal complement $V^\perp$, i.e., find a basis for $V^\perp$.

6) Given a subspace $V \subseteq \mathbf{R}^n$, find matrices for $\mathbf{Proj}_V$ (orthogonal projection onto the subspace V) and $\mathbf{Ref}_V$ (reflection through the subspace V).

7) Given two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbf{R}^n$, find the angle between $\mathbf{x}$ and $\mathbf{y}$ (using the standard inner product for $\mathbf{R}^n$).

8) Given an arbitrary basis $\{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ for a subspace $V \subseteq \mathbf{R}^n$, find an orthonormal basis for V using the Gram-Schmidt process.

9) Define in geometric terms what it means for a linear transformation T: $\mathbf{R}^n \to \mathbf{R}^n$ to be an *orthogonal transformation*. If $\mathbf{A}$ is the orthogonal matrix for this transformation, what special properties does $\mathbf{A}$ have?

10) How do you find the inverse of an orthogonal matrix?

11) Find a least-squares solution to the linear system $\mathbf{Ax} = \mathbf{b}$. Describe what this means geometrically.

12) Use the method of least-squares to find the best fitting linear relation to a given set of data.

13) Given an $n \times n$ matrix $\mathbf{A}$, find the determinant of $\mathbf{A}$ by hand (using patterns, row operations, Lagrange expansion, or whatever other correct method suits your fancy).

14) Describe some of the important properties of determinants, e.g.

a) det ($\mathbf{A}^T$) = det ($\mathbf{A}$)
b) linearity in the rows (or columns)
c) the effect of row operations on the value of the determinant
d) det ($\mathbf{AB}$) = (det $\mathbf{A}$) (det $\mathbf{B}$) for $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$
e) $\det(\mathbf{A}^{-1}) = \dfrac{1}{\det(\mathbf{A})}$ (for an invertible matrix $\mathbf{A}$).

15) Use Cramer's Rule to solve an invertible 2×2 or 3×3 linear system.

16) Use determinants to calculate areas, volumes, k-volumes in $\mathbf{R}^n$.

17) Find the eigenvalues and eigenvectors of an $n \times n$ matrix $\mathbf{A}$ and the algebraic and geometric multiplicities of each eigenvalue.

18) Determine whether an $n \times n$ matrix $\mathbf{A}$ is diagonalizable (over the reals), i.e. if it yields a basis of eigenvectors. In this case, find the powers $\mathbf{A}^t$ of this matrix and use this to find closed formulas and trajectories for the discrete linear dynamical system $\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t)$.

Math S-21b – Summer 2005 – Practice Exam #2

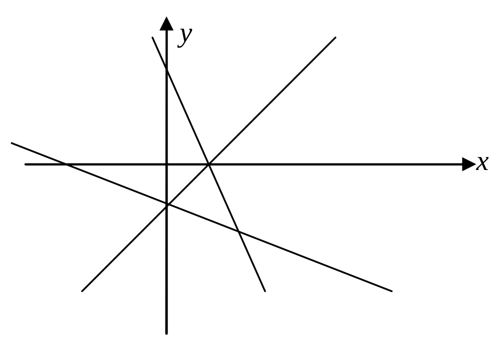
1) True or False. (Circle one) You need not give your reasoning.

a) A linear transformation T: $\mathbf{R}^n \to \mathbf{R}^n$ is an orthogonal transformation if and only if $T(\mathbf{x}) \cdot T(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for all vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbf{R}^n$.	TRUE FALSE
b) Let $\mathbf{A}$ be a square matrix with exactly one entry 1 in each row and in each column, the other entries being zero. Then $\mathbf{A}$ is an orthogonal matrix.	TRUE FALSE
c) If $\mathbf{A}$ is an $n \times n$ matrix, then det(2$\mathbf{A}$) = 2 (det $\mathbf{A}$).	TRUE FALSE
d) If 0 is an eigenvalue of the matrix $\mathbf{A}$, then det($\mathbf{A}$) = 0.	TRUE FALSE
e) If an $n \times n$ matrix $\mathbf{A}$ is diagonalizable, then it has n distinct eigenvalues.	TRUE FALSE
f) If $\mathbf{v}$ is a unit (column) vector in $\mathbf{R}^3$, then the matrix $\mathbf{v}\mathbf{v}^T$ is diagonalizable.	TRUE FALSE
g) If $\mathbf{A}$ is an $n \times n$ matrix, then $\mathbf{A}$ and $\mathbf{A}^T$ have the same eigenvalues.	TRUE FALSE

2) Consider the following <u>inconsistent</u> system of linear equations: $\begin{Bmatrix} 2x + y = 2 \\ x - y = 1 \\ x + 2y = -2 \end{Bmatrix}$.

Each equation represents a line in $\mathbf{R}^2$.

Find the least-squares solution for this linear system. Indicate your solution in the diagram (right) and briefly describe what you think the relationship is between this least-squares solution and the lines represented by the given linear equations.



3) We are given three vectors in $\mathbf{R}^4$: $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$.

a) Find the area of the parallelogram determined by the vectors $\{ \mathbf{v}_1 , \mathbf{v}_2 \}$.

b) Construct an orthonormal basis for the two-dimensional subspace of $\mathbf{R}^4$ spanned by $\{\mathbf{v}_1, \mathbf{v}_2\}$.
Call the vectors of this orthonormal basis $\mathbf{w}_1$, and $\mathbf{w}_2$.

c) Find the orthogonal projection of $\mathbf{v}_3$ in the subspace spanned by the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

d) If we let $\mathbf{B} = \begin{bmatrix} \uparrow & \uparrow \\ \mathbf{w}_1 & \mathbf{w}_2 \\ \downarrow & \downarrow \end{bmatrix}$ where $\{\mathbf{w}_1, \mathbf{w}_2\}$ is the orthonormal basis found in part b, what are the values of $\det(\mathbf{B}\mathbf{B}^T)$ and $\det(\mathbf{B}^T\mathbf{B})$? [*Hint*: You don't need to know what $\mathbf{w}_1$ and $\mathbf{w}_2$ are to calculate these two numbers.]

4) a) Find the eigenvalues and eigenvectors for the matrix $\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 6 & -4 & -4 \\ -6 & 6 & 6 \end{bmatrix}$. Show your work.

b) Calculate the vector $\mathbf{A}^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ for any power t.

[Your answer should be a vector whose components are functions of t.]