Math S-21b – Summer 2005 – Practice Exam #2 Solutions

1) True or False. (Circle one) You need not give your reasoning.

a) A linear transformation T: $\mathbf{R}^n \to \mathbf{R}^n$ is an orthogonal transformation if and only if $T(\mathbf{x})\bullet T(\mathbf{y}) = \mathbf{x}\bullet\mathbf{y}$ for all vectors **x** and **y** in $\mathbf{R}^n$.

TRUE – $T(\mathbf{x})\cdot T(\mathbf{x}) = \mathbf{x}\cdot\mathbf{x} \quad\Rightarrow\quad \|T(\mathbf{x})\|^2 = \|\mathbf{x}\|^2 \quad\Rightarrow\quad \|T(\mathbf{x})\| = \|\mathbf{x}\|$

b) Let **A** be a square matrix with exactly one entry 1 in each row and in each column, the other entries being zero. Then **A** is an orthogonal matrix.

TRUE – The columns will be orthogonal and will form a basis for $\mathbf{R}^n$.

c) If **A** is an $n \times n$ matrix, then det(2**A**) = 2 (det **A**).

FALSE – The determinant is linear in any given column, but if you scale all entries by 2, the resulting determinant will be scaled by 2^n, not by 2.

d) If 0 is an eigenvalue of the matrix **A**, then det(**A**) = 0.

TRUE – *Reason* 1: det **A** is equal to the product of its eigenvalues, hence det **A** = 0 if even one eigenvalue is 0.
Reason 2: If $\lambda = 0$ is an eigenvalue, and if **v** is a corresponding eigenvector, then $\mathbf{A}\mathbf{v} = \lambda\mathbf{v} = \mathbf{0}$. So
$\mathbf{v} \in \ker \mathbf{A} \quad\Rightarrow\quad \mathbf{A}$ is not invertible $\quad\Rightarrow\quad \det \mathbf{A} = 0$.

e) If an $n \times n$ matrix **A** is diagonalizable, then it has n distinct eigenvalues.

FALSE – **A** could have repeated eigenvalues with geometric multiplicity equal to the algebraic multiplicity, e.g. the identity matrix.

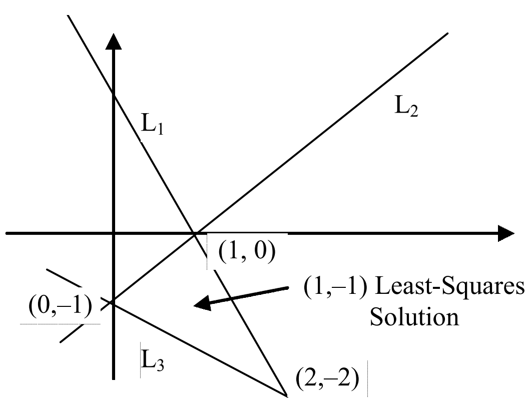
f) If **v** is a unit (column) vector in $\mathbf{R}^3$, then the matrix $\mathbf{v}\mathbf{v}^T$ is diagonalizable.

TRUE – The matrix $\mathbf{v}\mathbf{v}^T$ is the matrix for orthogonal projection onto span$\{\mathbf{v}\}$. Orthogonal projections are diagonalizable with two eigenvalues $\{0,1\}$, with E_1 = image of the projection, and with E_0 = orthogonal complement of the image of the projection.

g) If **A** is an $n \times n$ matrix, then **A** and $\mathbf{A}^T$ have the same eigenvalues.

TRUE - $\det(\lambda\mathbf{I} - \mathbf{A}^T) = \det(\lambda\mathbf{I} - \mathbf{A})$ since $\lambda\mathbf{I} - \mathbf{A}^T$ and $\lambda\mathbf{I} - \mathbf{A}$ are transposes of each other and $\det(\mathbf{B}^T) = \det(\mathbf{B})$.

2) Consider the following <u>inconsistent</u> system of linear equations: $\begin{Bmatrix} 2x + y = 2 \\ x - y = 1 \\ x + 2y = -2 \end{Bmatrix}$. Each equation represents a line in $\mathbf{R}^2$.



Find the least-squares solution for this linear system. Indicate your solution in the above diagram and briefly describe what you think the relationship is between this least-squares solution and the lines represented by the given linear equations.

SOLUTION: The matrix form of the original linear system is $\mathbf{Ax} = \mathbf{b}$ or $\begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$. The normal equation is $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$ or $\begin{bmatrix} 6 & 3 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$. The solution to this is $(x, y) = (1,-1)$.

The least-squares solution is clearly within the triangle formed by the three lines. It is at or near the geometric center of the triangle.

3) We are given three vectors in $\mathbf{R}^4$: $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$.

a) Find the area of the parallelogram determined by the vectors { $\mathbf{v}_1$, $\mathbf{v}_2$ }.

SOLUTION: $\mathbf{A} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \\ -1 & 0 \end{bmatrix}$ and $\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$, so det($\mathbf{A}^T\mathbf{A}$) = 8 – 4 = 4.

Therefore the area is $\sqrt{\det(\mathbf{A}^T\mathbf{A})} = \sqrt{4} = 2$.

b) Construct an orthonormal basis for the two-dimensional subspace of $\mathbf{R}^4$ spanned by {$\mathbf{v}_1$, $\mathbf{v}_2$}. Call the vectors of this orthonormal basis $\mathbf{w}_1$, and $\mathbf{w}_2$.

SOLUTION: $\|\mathbf{v}_1\| = \sqrt{1+1+1+1} = 2$, so $\mathbf{w}_1 = \dfrac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \dfrac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$. $\text{Proj}_{\mathbf{w}_1}(\mathbf{v}_2) = (\mathbf{v}_2 \cdot \mathbf{w}_1)\mathbf{w}_1 = \dfrac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$, so

$$\mathbf{v}_2 - \text{Proj}_{\mathbf{w}_1}(\mathbf{v}_2) = (\mathbf{v}_2 \cdot \mathbf{w}_1)\mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{w}_2$$, since it's already a unit vector. [Check.]

So $\{\mathbf{w}_1, \mathbf{w}_2\} = \left\{ \dfrac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \dfrac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is the orthonormal basis derived using the Gram-Schmidt process.

c) Find the orthogonal projection of $\mathbf{v}_3$ in the subspace spanned by the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

SOLUTION: If V = Span{$\mathbf{v}_1$, $\mathbf{v}_2$} = Span{$\mathbf{w}_1$, $\mathbf{w}_2$}, then

$$\text{Proj}_V(\mathbf{v}_3) = (\mathbf{v}_3 \cdot \mathbf{w}_1)\mathbf{w}_1 + (\mathbf{v}_3 \cdot \mathbf{w}_2)\mathbf{w}_2 = \tfrac{1}{2}(1+2+3+0)\tfrac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} + \tfrac{1}{2}(1-2+3+0)\tfrac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{3}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ -1 \end{bmatrix}$$

d) If we let $\mathbf{B} = \begin{bmatrix} \uparrow & \uparrow \\ \mathbf{w}_1 & \mathbf{w}_2 \\ \downarrow & \downarrow \end{bmatrix}$ where {$\mathbf{w}_1$, $\mathbf{w}_2$} is the orthonormal basis found in part b, what are the values of det($\mathbf{B}\mathbf{B}^T$) and det($\mathbf{B}^T\mathbf{B}$)? [*Hint*: You don't need to know what $\mathbf{w}_1$ and $\mathbf{w}_2$ are to calculate these two numbers.]

SOLUTION: The matrix $\mathbf{B}\mathbf{B}^T$ is the matrix for orthogonal projection onto the subspace V = Span{$\mathbf{w}_1$, $\mathbf{w}_2$}. This transformation is not invertible, so det($\mathbf{B}\mathbf{B}^T$) = 0.

$\mathbf{B}^T\mathbf{B} = \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{bmatrix} \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$ since {$\mathbf{w}_1$, $\mathbf{w}_2$} are orthonormal, so det($\mathbf{B}\mathbf{B}^T$) = det($\mathbf{I}$) = 1.

4) a) Find the eigenvalues and eigenvectors for the matrix $\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 6 & -4 & -4 \\ -6 & 6 & 6 \end{bmatrix}$. Show your work.

SOLUTION: $\lambda\mathbf{I} - \mathbf{A} = \begin{bmatrix} \lambda-2 & 0 & 0 \\ -6 & \lambda+4 & 4 \\ 6 & -6 & \lambda-6 \end{bmatrix}$, so $\det(\lambda\mathbf{I} - \mathbf{A}) = (\lambda-2)(\lambda^2 - 2\lambda) = (\lambda-2)^2\lambda = 0$ which gives the two eigenvalues $\lambda = 2$ (multiplicity 2) and $\lambda = 0$.

$$\lambda_1 = \lambda_2 = 2 \quad \Rightarrow \quad \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -6 & 6 & 4 & 0 \\ 6 & -6 & -4 & 0 \end{array}\right] \quad \Rightarrow \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$

$$\lambda_3 = 0 \quad \Rightarrow \quad \left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ -6 & 4 & 4 & 0 \\ 6 & -6 & -6 & 0 \end{array}\right] \quad \Rightarrow \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

b) Calculate the vector $\mathbf{A}^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ for any power t. [Your answer should be a vector whose components are functions of t.]

SOLUTION: If we write $\mathbf{S} = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & -1 \\ 0 & 3 & 1 \end{bmatrix}$, the change of basis matrix, we can compute $\mathbf{S}^{-1} = \begin{bmatrix} 3 & -2 & -2 \\ -1 & 1 & 1 \\ 3 & -3 & -2 \end{bmatrix}$.

We know that $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $\mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$ and $\mathbf{A}^t = \mathbf{S}\mathbf{D}^t\mathbf{S}^{-1}$, so we have

$$\mathbf{A}^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & -1 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2^t & 0 & 0 \\ 0 & 2^t & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & -2 & -2 \\ -1 & 1 & 1 \\ 3 & -3 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2^t & 2\cdot 2^t & 0 \\ 2^t & 0 & 0 \\ 0 & 3\cdot 2^t & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2^t \\ 3\cdot 2^t \\ -3\cdot 2^t \end{bmatrix} = 2^t \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}$$

Note: The vector $\begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}$ is in the eigenspace E_2, as expected.