Math S-21b Practice Final Exam Solutions – Summer 2005

The topics, difficulty level, and number of questions may be different on the actual exam.

1) **True/False** (circle one)

a) If **A** and **B** are $n \times n$ matrices such that **AB** = **BA**, and if **v** is an eigenvector of **A**, then **Bv** is also an eigenvector of **A**.

TRUE: **ABv** = **BAv** = **B**(λ**v**) = λ**Bv**, so **Bv** will be an eigenvector of **A** with the same eigenvalue λ.

b) If **A** is not a square matrix, then $\mathbf{AA}^T$ is not invertible.

FALSE: For example, if **A** is a 1×2 matrix like [1 2], then $\mathbf{AA}^T = [5]$, an invertible 1×1 matrix.

c) If **A** is a real 5×4 matrix, then $\mathbf{AA}^T$ is positive definite.

FALSE: For example, if the columns of **A** are orthonormal, then $\mathbf{AA}^T$ will be the matrix for projection onto the 4-dimensional subspace spanned by those columns. In this case, one of the eigenvalues would be 0. It's also true that the rank of **A** and $\mathbf{A}^T$ is at most 4 and the same must be true for $\mathbf{AA}^T$, so it's not full rank and cannot be invertible.

d) If the columns of an $m \times n$ matrix **A** are linearly independent, then the columns of its transpose $\mathbf{A}^T$ will be linearly independent as well.

FALSE: If $m > n$ and the n columns are linearly independent, then the rank of **A** will be n. The rank of $\mathbf{A}^T$ will also be n, so the dimension of the image of $\mathbf{A}^T$ will be n. Hence the m columns of $\mathbf{A}^T$ cannot all be linearly independent.

e) For all real numbers c the matrix $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ c & 0 & 1 \end{bmatrix}$ is invertible.

FALSE: The determinant of this matrix is $c + 1$, so when $c = -1$, the determinant will be 0 and the matrix will not be invertible.

f) If **v** is a unit (column) vector in $\mathbf{R}^3$, then the matrix $\mathbf{vv}^T$ is diagonalizable.

TRUE: This matrix (which represents orthogonal projection on Span(**v**)) is symmetric, so not only is it diagonalizable, it's orthogonally diagonalizable.

g) If two matrices have the same characteristic polynomial, then they have the same rank.

FALSE: This one's pretty subtle, but it's possible to have two matrices with the same characteristic polynomial and the same eigenvalues where one of them has algebraic multiplicity greater than 1 but the geometric multiplicities of that eigenvalue are different for the two matrices. In particular, if that repeated eigenvalue is 0, then $\dim(E_0)$, i.e. the dimensional of the kernel, will be different for the two matrices. Since the dimension of the kernel plus the dimension of the image, i.e. the rank, will sum to the same value, the ranks of the two matrices will in this case be different.

(h) Any symmetric 2×2 matrix has two distinct eigenvalues.

FALSE: Any symmetric matrix will be diagonalizable, but the multiplicity of any one eigenvalue need not be one. For example, the 2×2 identity matrix is diagonal and it has eigenvalue 1 with multiplicity 2, i.e. the eigenvalues are not distinct.

2) Short answer questions:

a) Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$. Calculate the area of the parallelogram formed by **u** and **v**.

Solution: Let $\mathbf{A} = \begin{bmatrix} \uparrow & \uparrow \\ \mathbf{u} & \mathbf{v} \\ \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ -1 & 2 \\ 3 & 0 \end{bmatrix}$. Then $\mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{bmatrix} 15 & 0 \\ 0 & 5 \end{bmatrix}$ and Area = $\sqrt{\det(\mathbf{A}^{\mathrm{T}}\mathbf{A})} = \sqrt{75} = 5\sqrt{3}$.

b) Find the matrix representing the linear transformation from $\mathbf{R}^2$ to $\mathbf{R}^2$ that is reflection in the line spanned by the vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

Solution: There are many good approaches to this one. For example, using the orthonormal eigenbasis $\mathcal{B} = \left\{ \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$, the matrix relative to this basis will be $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. From this we can work backwards to deduce that the matrix relative to the standard basis is $\frac{1}{5}\begin{bmatrix} -3 & -4 \\ -4 & 3 \end{bmatrix}$.

c) Find a matrix with eigenvalues equal to 2,3,5,7.

Solution: Don't look for anything subtle. The matrix $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$ will do just fine.

d) Consider the vector space V consisting of all 2×2 matrices for which the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector. Find a basis for this space, and determine its dimension.

Solution: If we write any such matrix as $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the given condition becomes:

$$\mathbf{A}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} a+2b \\ c+2d \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \lambda \\ 2\lambda \end{bmatrix}.$$

From this we deduce that $a = \lambda - 2b$ and $c = 2\lambda - 2d$. Substituting these into the matrix, we have that:

$$\mathbf{A} = \begin{bmatrix} \lambda - 2b & b \\ 2\lambda - 2d & d \end{bmatrix} = \lambda \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} + b \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}.$$

It's easy to show that these three matrices are linearly independent, so they form a basis for V and the dimension is therefore 3.

3) Consider a linear transformation $\mathrm{T}: \mathbf{R}^2 \to \mathbf{R}^2$. Suppose the matrix of T with respect to the basis $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$ is $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Find the matrix of T with respect to the basis $\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\}$.

Solution: For the first basis, the change of basis matrix is $\mathbf{S} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$, so $\mathbf{S}^{-1}\mathbf{AS} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. From this we get that $\mathbf{A} = \begin{bmatrix} 6 & -2 \\ 13 & -4 \end{bmatrix}$. For the second basis, the change of basis matrix is $\mathbf{P} = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$ and the matrix of this transformation relative to the second basis is given by $\mathbf{P}^{-1}\mathbf{AP} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 6 & -2 \\ 13 & -4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -1 & -5 \\ 1 & 3 \end{bmatrix}$.

4) Let **A** be a real $n \times n$ matrix such that $\mathbf{A}^2 = -\mathbf{I}_n$.

a) Show that **A** is invertible.

Solution: One way to do this is to note that $\mathbf{A}(-\mathbf{A}) = -\mathbf{A}^2 = \mathbf{I}$, so $-\mathbf{A}$ is the inverse of **A**.

b) Show that n must be even.

Solution: $(\det \mathbf{A})^2 = \det(\mathbf{A}^2) = \det(-\mathbf{I}) = (-1)^n$. Since $(\det \mathbf{A})^2$ is positive, it must be the case that n is even.

c) Show that **A** has no real eigenvalues.

Solution: Suppose **v** is an eigenvector of **A** with eigenvalue λ. Then $\mathbf{Av} = \lambda\mathbf{v}$ and $\mathbf{A}^2\mathbf{v} = \lambda^2\mathbf{v} = \lambda^2\mathbf{v} = -\mathbf{v}$. Therefore $(\lambda^2 + 1)\mathbf{v} = \mathbf{0}$. This can only be the case if $\lambda^2 + 1 = 0$, so the only eigenvalues of **A** are $\pm\, i$.

5) Consider the quadratic form $q(x_1, x_2, x_3) = 2x_1^{\ 2} + 2x_2^{\ 2} + 2x_3^{\ 2} - 2x_1x_2 - 2x_2x_3 + 2x_1x_3$.

a) Find a symmetric matrix **A** such that $q(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{Ax}$ for all $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in $\mathbf{R}^3$.

b) Find all the eigenvalues of **A** and their algebraic and geometric multiplicities.

c) Is **A** positive definite? Briefly justify your answer.

d) Find an orthonormal eigenbasis for **A**.

Solution: The matrix we want is $\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$. Calculation gives the eigenvalues {4, 1, 1}. By the Spectral Theorem, we know that **A** is orthogonally diagonalizable, so AM(4) = GM(4) = 1 and AM(1) = GM(1) = 2. Since all eigenvalues are positive, the matrix is positive definite. An orthonormal eigenbasis is $\left\{ \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\}$.

6) A rabbit population and a wolf population are modeled by the equations

$$r(t+1) = 5r(t) - 2w(t)$$
$$w(t+1) = r(t) + 2w(t)$$

The initial populations are $r(0)=300$ and $w(0)=200$.

a) Find formulas for $r(t)$ and $w(t)$.

b) In the long run, what will be the proportion of rabbits to wolves? Explain.

Solution: If we let $\mathbf{x} = \begin{bmatrix} r \\ w \end{bmatrix}$, this system can be expressed as $\mathbf{x}(t+1) = \mathbf{Ax}(t)$ where $\mathbf{A} = \begin{bmatrix} 5 & -2 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{x}(0) = \begin{bmatrix} 300 \\ 200 \end{bmatrix}$. The solution is $\mathbf{x}(t) = \mathbf{A}^t\mathbf{x}(0)$. The eigenvalues of **A** are 4 and 3 and their respective eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The change of basis matrix is $\mathbf{S} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{S}^{-1}\mathbf{AS} = \mathbf{D} = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$.

From this we calculate that $\mathbf{A} = \mathbf{SDS}^{-1}$ and $\mathbf{A}^t = \mathbf{SD}^t\mathbf{S}^{-1}$, so

$$\mathbf{x}(t) = \mathbf{SD}^t\mathbf{S}^{-1}\mathbf{x}(0) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4^t & 0 \\ 0 & 3^t \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 300 \\ 200 \end{bmatrix} = \begin{bmatrix} 2\cdot 4^t & 3^t \\ 4^t & 3^t \end{bmatrix}\begin{bmatrix} 100 \\ 100 \end{bmatrix} = 100\cdot 4^t\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 100\cdot 3^t\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

From this we can see that $r(t) = 200\cdot 4^t + 100\cdot 3^t$ and $w(t) = 100\cdot 4^t + 100\cdot 3^t$.

In the long run, the dominant eigenvalue will rule and the proportion of rabbits to wolves will approach that given by the components of the dominant eigenvector, namely 2:1.

7) Let $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -b & -c \end{bmatrix}$ where b and c are real numbers.

Consider the continuous dynamical system $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$.

a) What inequality or inequalities involving b and c ensure that the solutions to the system will consist of trajectories spiraling inwards toward the origin?

Solution: The eigenvalues are the roots of $\lambda^2 + c\lambda + b = 0$. This gives $\lambda = \dfrac{-c \pm \sqrt{c^2 - 4b}}{2}$.

These roots will be imaginary when $c^2 - 4b < 0$. The real part will be negative when c is positive and this corresponds to a spiral in. In contrast, the trajectories will spiral out when c is negative.
The roots will be real when $c^2 - 4b > 0$. In that case, there are three possibilities: two positive eigenvalues (source), two negative eigenvalues (sink), or one positive and one negative eigenvalue (one stable and one unstable direction).

b) Solve this continuous dynamical system in the case where $b = 4$, $c = 5$, and $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$**.** (Your answer should be a closed formula for $\mathbf{x}(t)$.)

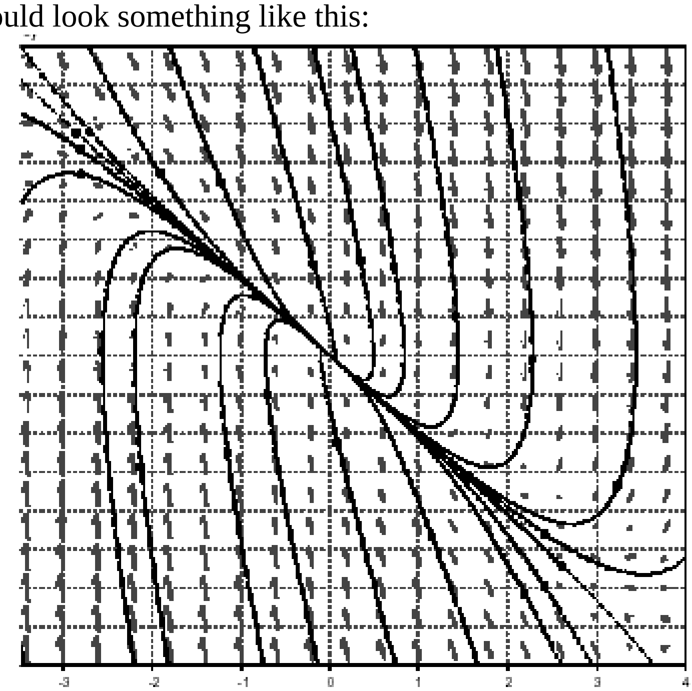
Solution: Here we have $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix}$. This gives $\lambda\mathbf{I} - \mathbf{A} = \begin{bmatrix} \lambda & -1 \\ 4 & \lambda + 5 \end{bmatrix}$. The eigenvalues are the roots of $\lambda^2 + 5\lambda + 4 = (\lambda + 1)(\lambda + 4) = 0$. This yields the two negative eigenvalues –4 and –1 (a sink). The eigenvectors are, respectively, $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and the change of basis matrix is $\mathbf{S} = \begin{bmatrix} -1 & 1 \\ 4 & -1 \end{bmatrix}$. The solution is given by:

$$\mathbf{x}(t) = [e^{t\mathbf{A}}]\mathbf{x}(0) = \mathbf{S}[e^{t\mathbf{D}}]\mathbf{S}^{-1}\mathbf{x}(0) = \begin{bmatrix} -1 & 1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} e^{-4t} & 0 \\ 0 & e^{-t} \end{bmatrix} \tfrac{1}{3} \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -e^{-4t} & e^{-t} \\ 4e^{-4t} & -e^{-t} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = e^{-4t} \begin{bmatrix} -1 \\ 4 \end{bmatrix} + 3e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -e^{-4t} + 3e^{-t} \\ 4e^{-4t} - 3e^{-t} \end{bmatrix}$$

c) Sketch the phase portrait for this continuous dynamical system.

Solution: Your sketch should look something like this:



8) Consider the system $\left\{ \begin{aligned} \frac{dx}{dt} &= x - y \\ \frac{dy}{dt} &= x^2 - y \end{aligned} \right\}$.

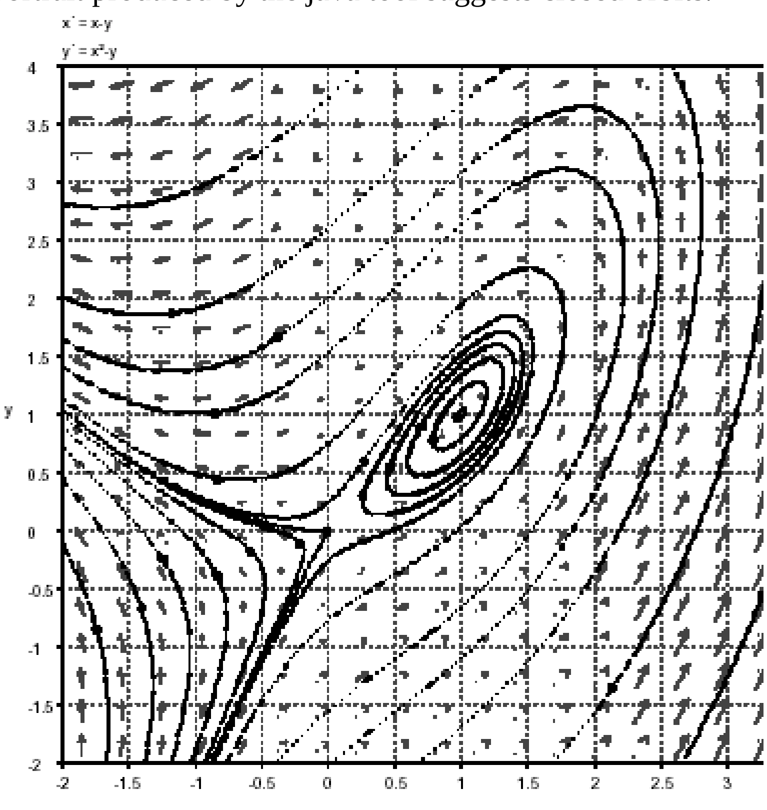
a) Perform the qualitative phase plane analysis for this system (i.e., find the null clines, equilibrium points, and general directions). Carry this out in the whole *xy*-plane (not just the first quadrant).

b) List the equilibrium points of the system above, and determine their stability. That is, linearize the system at each equilibrium and do the eigenvalue-eigenvector analysis. Give a rough sketch of some solutions, particularly in the vicinity of the equilibria.

Solution: The vertical nullcline is the line $y = x$, and the horizontal nullcline is the parabola $y = x^2$. There are equilibria at the points (0, 0) and (1, 1). A picture of the nullclines, equilibria, and trajectories is given below. For the equilibrium analysis, we look at the Jacobian matrix $\mathbf{J}_F = \begin{bmatrix} 1 & -1 \\ 2x & -1 \end{bmatrix}$.

At the (0, 0) equilbrium, we have $\mathbf{J}_F(0,0) = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$. This yields eigenvalues +1 and –1 and respective eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and the equilibrium will have one unstable and one stable direction.

At the (1, 1) equilibrium, we have $\mathbf{J}_F(1,1) = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$. This yields the complex eigenvalues $\pm i$. If the system were linear, this would indicate closed trajectories encircling the equilibrium. However, since this is a nonlinear system, this linear approximation in the vicinity of the equilibrium is inconclusive. It could yield closed orbits, spirals slowly converging to the equilibrium, or spirals slowly diverging away from the equilibrium. The phase portrait produced by the java tool suggests closed orbits.



9) a) Find all solutions of the differential equation $f'' - f' - 6f = 6t$.

b) Find a solution which satisfies $f'(0) = f''(0) = 30$. Is such a solution unique?

Solution: We have to find the solution to the corresponding homogeneous equation, then find a particular solution. The general solution will be the sum of these. For the homogeneous solution, we solve $f'' - f' - 6f = 0$. The characteristic equation is $\lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2) = 0$ which yields the roots 3 and 2. This gives the homogeneous solution $f_h(t) = c_1 e^{3t} + c_2 e^{-2t}$.

For a particular solution, try a solution of the form $f_p(t) = At + B$. Substitution this into the inhomogeneous equation, we get $-6At - (A + 6B) = 6t$. This gives us $A = -1$ and $B = \frac{1}{6}$, so the particular solution is $f_p(t) = -t + \frac{1}{6}$, and the general solution is $f(t) = c_1 e^{3t} + c_2 e^{-2t} - t + \frac{1}{6}$.

Finally, we take the first and second derivatives, substitute the given initial conditions, and find the coefficients. Ultimately, this gives the unique solution $f(t) = \frac{1}{30}(184e^{3t} - 189e^{-2t}) - t + \frac{1}{6}$.

BONUS QUESTION

10) Consider the two subspaces V_1 and V_2 of $\mathbf{R}^4$, where $V_1 = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix} \right\}$ and $V_2 = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Find a basis for the intersection $V_1 \cap V_2$. [Note: The intersection of two subspaces is also a subspace.]

Solution: This requires some thought, but the basic idea is to focus on orthogonal complements. A little work yields that a basis for $V_1^\perp$ is $\begin{bmatrix} -19 \\ 11 \\ -1 \\ 7 \end{bmatrix}$, and a basis for $V_2^\perp$ is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$. A vector is in the intersection of V_1 and V_2 if it is orthogonal to all three of these vectors, i.e. in the kernel of $\begin{bmatrix} -19 & 11 & -1 & 7 \\ -2 & 1 & 1 & 0 \\ 3 & -2 & 0 & 1 \end{bmatrix}$. After a little row reduction, we conclude that a basis for this subspace is the vector $\begin{bmatrix} 13 \\ 21 \\ 5 \\ 3 \end{bmatrix}$. Note that this means that the intersection has dimension 1 in this case.