

Constructing Curves and Surfaces through Specified Points

A technique for using determinants to construct lines, circles, and general conic sections through specified points in the plane is described. The procedure is also used to pass planes and spheres in three-dimensional space through fixed points.

PREREQUISITES:

Linear systems
Determinants
Analytic geometry

INTRODUCTION

One of the fundamental results in the theory of Linear Algebra is the following:

A homogeneous linear system with as many equations as unknowns has a nontrivial solution if and only if the determinant of the system is zero.

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In this chapter, we show how this result may be used to determine the equations of various curves and surfaces through specified points. We proceed immediately to some specific examples.

A LINE THROUGH TWO POINTS

Suppose we are given two distinct points in the plane, (x_1, y_1) and (x_2, y_2) . There is a unique line,

$$c_1x + c_2y + c_3 = 0, \quad (1.1)$$

which passes through these two points. Notice that c_1 , c_2 , and c_3 are not all zero, and that these coefficients are unique only up to a multiplicative constant.

Since (x_1, y_1) and (x_2, y_2)

lie on the line, substituting them in (1.1) gives the two equations

$$c_1x_1 + c_2y_1 + c_3 = 0 \quad (1.2)$$

$$c_1x_2 + c_2y_2 + c_3 = 0. \quad (1.3)$$

The three equations, (1.1), (1.2), and (1.3), may be grouped together in system form as

$$x c_1 + y c_2 + c_3 = 0$$

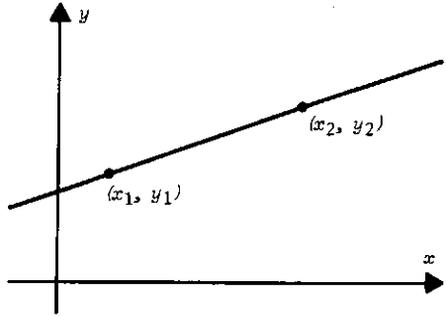
$$x_1 c_1 + y_1 c_2 + c_3 = 0$$

$$x_2 c_1 + y_2 c_2 + c_3 = 0.$$

In this form, we have a homogeneous system of three equations for c_1 , c_2 , and c_3 . Since c_1 , c_2 , and c_3 are not all zero, this system has a nontrivial solution, and so the determinant of the system must be zero. That is,

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0. \quad (1.4)$$

Consequently, every point (x, y) on the line satisfies (1.4), and conversely, every point (x, y) which satisfies (1.4) lies on the line.



EXAMPLE 1.1 Find the equation of the line which passes through the two points $(2, 1)$ and $(3, 7)$.

SOLUTION Substitution of the coordinates of the two points into Eq. (1.4) gives

$$\begin{vmatrix} x & y & 1 \\ 2 & 1 & 1 \\ 3 & 7 & 1 \end{vmatrix} = 0.$$

The cofactor expansion of this determinant along the first row then gives:

$$-6x + y + 11 = 0.$$

A CIRCLE THROUGH THREE POINTS

Let us be given three distinct points in the plane, (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) , not all lying on a straight line. From analytic geometry, we know that there is a unique circle, say

$$c_1(x^2 + y^2) + c_2x + c_3y + c_4 = 0, \quad (1.5)$$

which passes through them (Fig. 1.2). Substituting the coordinates of the three points into this equation gives

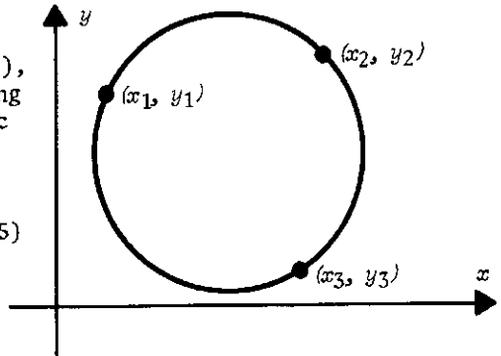


Figure 1.2

$$c_1(x_1^2 + y_1^2) + c_2x_1 + c_3y_1 + c_4 = 0 \quad (1.6)$$

$$c_1(x_2^2 + y_2^2) + c_2x_2 + c_3y_2 + c_4 = 0 \quad (1.7)$$

$$c_1(x_3^2 + y_3^2) + c_2x_3 + c_3y_3 + c_4 = 0. \quad (1.8)$$

As before, Eqs. (1.5) - (1.8) form a homogeneous linear system with a nontrivial solution for c_1 , c_2 , c_3 , and c_4 . Thus the determinant of this linear system is zero:

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$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0. \quad (1.9)$$

This is a determinant form for the equation of the circle.

EXAMPLE 1.2 Find the equation of the circle which passes through the three points (1, 7), (6, 2), and (4, 6).

SOLUTION Substitution of the coordinates of the three points into Eq. (1.9) gives

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ 50 & 1 & 7 & 1 \\ 40 & 6 & 2 & 1 \\ 52 & 4 & 6 & 1 \end{vmatrix} = 0,$$

which reduces to

$$10(x^2 + y^2) - 20x - 40y - 200 = 0.$$

In standard form this is

$$(x - 1)^2 + (y - 2)^2 = 5^2.$$

Thus, the circle has center (1, 2) and radius 5.

A GENERAL CONIC SECTION THROUGH FIVE POINTS

The general equation of a conic section in the plane (a parabola, hyperbola, or ellipse, and degenerate forms of these three curves) is given by

$$c_1x^2 + c_2xy + c_3y^2 + c_4x + c_5y + c_6 = 0.$$

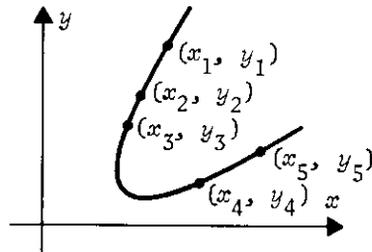


Figure 1.3

This form contains six coefficients, although only five are needed since we may divide through by any one of them which is not zero. Thus, only five coefficients must be determined, so that five distinct points in the plane are sufficient to determine the equation of the conic section (Fig. 1.3). As before, the equation may be put in determinant form (see Exercise 1.6):

$$\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{vmatrix} = 0. \quad (1.10)$$

EXAMPLE 1.3 An astronomer wants to determine the orbit of an asteroid about the sun. He sets up a Cartesian coordinate system in the plane of the orbit with the sun at the origin. Astronomical units of measurement are used along the axes. (1 astronomical unit = mean distance of earth to sun = 93 million miles.) By Kepler's first law, he knows that the orbit must be an ellipse. Consequently, he makes five observations of the asteroid at five different times and finds five points along the orbit to be

$$(5.764, 0.648), (6.286, 1.202), (6.759, 1.823), \\ (7.168, 2.526), (7.480, 3.360).$$

Find the equation of the orbit.

SOLUTION Substitution of the coordinates of the five given points into Eq. (1.10) gives

$$\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ 33.224 & 3.735 & 0.420 & 5.764 & 0.648 & 1 \\ 39.514 & 7.556 & 1.445 & 6.286 & 1.202 & 1 \\ 45.684 & 12.322 & 3.323 & 6.759 & 1.823 & 1 \\ 51.380 & 18.106 & 6.381 & 7.168 & 2.526 & 1 \\ 55.950 & 25.133 & 11.290 & 7.480 & 3.360 & 1 \end{vmatrix} = 0.$$

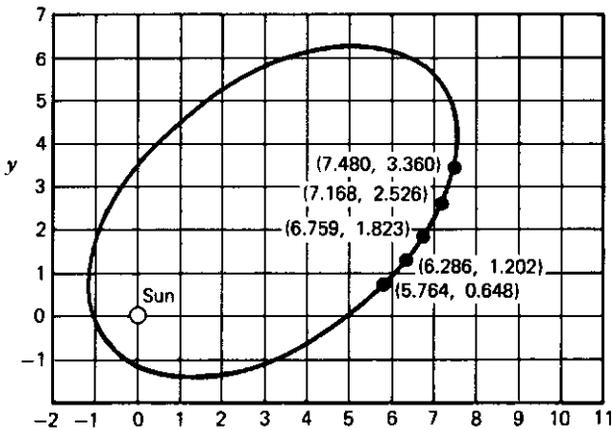


Figure 1.4

The cofactor expansion of this determinant along the first row reduces to

$$x^2 - 1.04xy + 1.30y^2 - 3.90x - 2.93y - 5.49 = 0.$$

Figure 1.4 is a diagram of the orbit, together with the five given points.

A PLANE THROUGH THREE POINTS

In Exercise 1.7 we ask the reader to show the following: The plane in 3-space with equation

$$c_1x + c_2y + c_3z + c_4 = 0$$

which passes through three noncollinear points (x_1, y_1, z_1) , (x_2, y_2, z_2) , and (x_3, y_3, z_3) is given by the determinant equation

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0. \quad (1.11)$$

EXAMPLE 1.4 The equation of the plane which passes through the three noncollinear points $(1, 1, 0)$, $(2, 0, -1)$, and $(2, 9, 2)$ is

$$\begin{vmatrix} x & y & z & 1 \\ 1 & 1 & 0 & 1 \\ 2 & 0 & -1 & 1 \\ 2 & 9 & 2 & 1 \end{vmatrix} = 0,$$

which reduces to

$$2x - y + 3z - 1 = 0.$$

A SPHERE THROUGH FOUR POINTS

In Exercise 1.8 we ask the reader to show the following: The sphere in 3-space with equation

$$c_1(x^2 + y^2 + z^2) + c_2x + c_3y + c_4z + c_5 = 0$$

which passes through four noncoplanar points (x_1, y_1, z_1) , (x_2, y_2, z_2) , (x_3, y_3, z_3) , and (x_4, y_4, z_4) is given by the following determinant equation:

$$\begin{vmatrix} x^2 + y^2 + z^2 & x & y & z & 1 \\ x_1^2 + y_1^2 + z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2 + y_2^2 + z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2 + y_3^2 + z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2 + y_4^2 + z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0. \quad (1.12)$$

EXAMPLE 1.5 The equation of the sphere which passes through the four points $(0, 3, 2)$, $(1, -1, 1)$, $(2, 1, 0)$, and $(-1, 1, 3)$ is

$$\begin{vmatrix} x^2 + y^2 + z^2 & x & y & z & 1 \\ 13 & 0 & 3 & 2 & 1 \\ 3 & 1 & -1 & 1 & 1 \\ 5 & 2 & 1 & 0 & 1 \\ 11 & -1 & 1 & 3 & 1 \end{vmatrix} = 0.$$

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This reduces to

$$x^2 + y^2 + z^2 - 4x - 2y - 6z + 5 = 0,$$

which in standard form is

$$(x - 2)^2 + (y - 1)^2 + (z - 3)^2 = 9.$$

EXERCISES

- 1.1** Find the equations of the lines which pass through the following points:
(a) (1, -1), (2, 2)
(b) (0, 1), (1, -1).
- 1.2** Find the equations of the circles which pass through the following points:
(a) (2, 6), (2, 0), (5, 3)
(b) (2, -2), (3, 5), (-4, 6).
- 1.3** Find the equation of the conic section which passes through the points (0, 0), (0, -1), (2, 0), (2, -5), and (4, -1).
- 1.4** Find the equations of the planes in 3-space which pass through the following points:
(a) (1, 1, -3), (1, -1, 1), (0, -1, 2)
(b) (2, 3, 1), (2, -1, -1), (1, 2, 1).
- 1.5** Find the equations of the spheres in 3-space which pass through the following points:
(a) (1, 2, 3), (-1, 2, 1), (1, 0, 1), (1, 2, -1)
(b) (0, 1, -2), (1, 3, 1), (2, -1, 0), (3, 1, -1).
- 1.6** Show that Eq. (1.10) is the equation of the conic section which passes through five given distinct points.
- 1.7** Show that Eq. (1.11) is the equation of the plane in 3-space which passes through three given noncollinear points.
- 1.8** Show that Eq. (1.12) is the equation of the sphere in 3-space which passes through four given noncoplanar points.
- 1.9** Find a determinant equation for the parabola of the form

$$c_1y + c_2x^2 + c_3x + c_4 = 0$$

which passes through three given noncollinear points in the plane.