

Mathematics 223a Homework due September 25

September 12, 2008

1 Fractional ideals in Dedekind Domains.

Given A a Dedekind Domain with fraction field K , recall that a *fractional ideal* of A is a nonzero finitely generated A -module $J \subset K$. If J_1, J_2 are two fractional ideals of A , then their *product* is defined as follows $J_1 \cdot J_2 = \{\sum_{i=1}^s x_i y_i \in K\}$ the elements ranging over $s \in \mathbf{N}$, $x_i \in J_1$ and $y_i \in J_2$.

1. The ideal $J_1 \cdot J_2$ is a fractional ideal of A .
2. If J is a fractional ideal of A , show that J is *locally principal* in the sense that for every prime ideal P of A , the localization of J at P , i.e., $J_P := J \cdot A_P \subset A_P$, is a principal ideal of A_P .
3. If J_1, J_2 are two fractional ideals of A , show that the natural A -homomorphism

$$J_1 \otimes_A J_2 \rightarrow J_1 \cdot J_2$$

that sends $x \otimes_A y$ to $x \cdot y \in K$ is an isomorphism of A modules.

4. Recall that two fractional ideals, J_1, J_2 , of A , are *equivalent* if there is an element $x \in K^*$ such that $x \cdot J_1 = J_2$. Show that two fractional ideals are equivalent if and only if they are isomorphic as A -modules.
5. If J is a fractional ideal, let

$$J^{-1} := \{x \in K \mid xJ \subset A\}.$$

Show that J^{-1} is a fractional ideal, and give a complete proof of the fact that $J \cdot J^{-1} = (1) = A$ (and hence that the set of fractional ideals forms a group).

Hint: Suppose that J is generated by elements $a_1, a_2, \dots, a_s \in A$. Show that $J \cdot J^{-1}$ is contained in no maximal ideal of A . Let $P \subset A$ be a nonzero prime ideal. Find an element $a \in J$ that is a generator of the ideal JA_P in the ring A_P (J is *locally principal*; why?) so that we have $JA_P = aA_P$, and in particular, $a^{-1}a_i A_P \subset A_P$ for all i ; i.e., there are elements $x_i \in A - P$ such that

$$\frac{a_i}{a} \in \frac{1}{x_i} \cdot A \subset K.$$

Now show that $\prod_a x_i \in J^{-1}$, and therefore $\prod x_i$ which is not in P lies in $J \cdot J^{-1}$.

6. More generally, for two fractional ideals I and J of A let $(I : J) = \{x \in K \mid xJ \subset I\}$. Show $(I : J)$ is a fractional ideal and $(I : J)J = I$. Give an A -module isomorphism $(I : J) \cong \text{Hom}_A(J, I)$.
7. For an ideal $I \subset A$ and a maximal ideal $P \subset A$, define $v_P(I) := \inf_{x \in I} v_P(x)$. Show that

$$I = \{x \in K \mid v_P(x) \geq v_P(I) \text{ for all } P\} = \prod_P P^{v_P(I)},$$

with negative exponents taken in the sense of the group formed by the fractional ideals of A .

2 Norm calculations.

- Factor $\frac{5+\sqrt{-23}}{2}$ into a product of prime ideals in the ring $\mathbb{Z}[\frac{1+\sqrt{-23}}{2}]$.
- Find an element of the ring $\mathbb{Z}[\frac{1+\sqrt{-23}}{2}]$ not contained in \mathbb{Z} that generates a prime ideal.

3 Invertibility; for you own edification: these problems you needn't hand in.

Let A be *any* commutative ring.

- By an *invertible* A -module, M let us mean an A -module for which there is an *inverse* in the sense of tensor products, i.e., such that there is an A -module N and an isomorphism of A -modules $M \otimes_A N \cong A$. *Note: You have shown, above, that fractional ideals in a Dedekind domain A are invertible A -modules, and their inverses are given by the construction above.*
- If A is an integral domain with field of fractions K and M an invertible A -module, show that the tensor product $M \otimes_A K$ is a K -vector space of dimension one. If A is a Dedekind Domain, show that any invertible A -module is isomorphic (as A -module) to a fractional ideal.
- For a general (commutative) ring A , show that invertible A -modules are flat. Show that they are finitely generated; *hint: let $\iota : M \otimes_A N \cong A$; find an equation $\iota \sum_{i=1}^s m_i \otimes n_i = 1 \in A$ and show that if $M_o :=$ the A -submodule of M generated by the m_i , then $M_o = M$ using flatness of N (and M).*
- Show that *the set of isomorphism classes of invertible A -modules* forms a group under the tensor product operation. This group is usually denoted $\text{Pic}(A)$, or $\text{Pic}(X)$ for $X = \text{Spec}A$. If A is a Dedekind Domain, the ideal class group of A is isomorphic to the group $\mathcal{H}(A)$ of equivalence classes of fractional ideals of A .