

GROUP COHOMOLOGY HOMEWORK 1

1. The transfer homomorphism. Let U be a group and V a subgroup of finite index. Let's denote the commutator subgroup of U by U^c and the maximal abelian quotient of U by $U^{\text{ab}} = U/U^c$. The transfer is a canonical homomorphism: $t_V^U : U^{\text{ab}} \rightarrow V^{\text{ab}}$ defined by a product over the cosets of V in U . For each right coset C let $\bar{C} \in C$ be a chosen representative so that $C = C\bar{C}$. Show that the map $u \mapsto \prod_C \bar{C}u\bar{C}u^{-1}$ induces a homomorphism $t_V^U : U^{\text{ab}} \rightarrow V^{\text{ab}}$ which is independent of the choice of coset representatives. Show also that if V is abelian and normal in U , then the image of t_V^U is contained in V^U , the intersection of V and the center of U .

2. Transfer in group extensions. Suppose G is a finite group, A a G -module written multiplicatively, and U a group extension of G by A . Let $u_\sigma \in U$ represent $\sigma \in G$ and $u_\sigma u_\tau = a_{\sigma,\tau} u_{\sigma\tau}$. Show:

$$t_A^U(u_\tau) = \prod_{\sigma \in G} a_{\sigma,\tau} \quad \text{and} \quad t_V^U(a) = N_G a := \prod_{\sigma \in G} a^\sigma.$$

3. The Nakayama map. With G and A as in **2.**, let $\alpha \in \mathcal{H}^2(G, A)$ and let $(\sigma, \tau) \mapsto a_{\sigma,\tau}$ be a 2-cocycle of G in A representing α . Show that the map $\tau \mapsto \prod_{\sigma \in G} a_{\sigma,\tau}$ induces a homomorphism $n_\alpha : G^{\text{ab}} \rightarrow A^G/N_G A$ (called Nakayama map) which depends only on α , not on the choice of cocycle representing α . Show also that $n_\alpha n_\beta = n_{\alpha\beta}$, so that the map $(\alpha, \sigma G^c) \mapsto n_\alpha(\sigma G^c)$ is bimultiplicative, i.e., is a “pairing” of $\mathcal{H}^2(G, A)$ and G^{ab} into $A^G/N_G A$.

4. The case G cyclic. Interpret **2.** and **3.** in case G is finite cyclic, and show that if σ a generates G , then the map $\alpha \mapsto n_\alpha(\sigma)$ is an isomorphism $\mathcal{H}^2(G, A) \xrightarrow{\sim} A^G/N_G A$.

5. $|G|$ kills $\mathcal{H}^r(G, A)$ for $r > 0$. a) Let $n = |G|$ be the order of G . Show $n\mathcal{H}^1(G, A) = 0$. [Hint: Sum $\sigma f(\tau) - f(\sigma\tau) + f(\sigma) = 0$ over $\tau \in G$.]

b) By ‘dimension shifting’ show that $n\mathcal{H}^r(G, A) = 0$ for all $r > 0$, and conclude in particular that if $\alpha \mapsto n\alpha$ is an automorphism of A , then $\mathcal{H}^r(G, A) = 0$ for all $r > 0$.

c) Show that $\mathcal{H}^2(G, \mathbb{Z}) = \mathcal{H}^1(G, \mathbb{C}^*) = \text{Hom}(G, \mathbb{C}^*)$. [Hint: Consider the exponential map $\mathbb{C} \rightarrow \mathbb{C}^*$.]

6. $\mathcal{H}^2(C_2 \times C_2, C_2)$. Let $A = \{1, \alpha\}$ be a group of order 2. Let $G = \{1, \rho, \sigma, \tau\}$ be a Klein four-group, i.e. a group isomorphic to $A \times A$. Considering group extensions U of G by A and the corresponding 2-cocycle classes α , show the following. For each vector $v = (a, b, c) \in A \times A \times A$ there is a unique class $\alpha_v \in \mathcal{H}^2(G, A)$ which has a representative standard 2-cocycle $a_{x,y}$ with $a_{1,1} = 1$ and $v = (a_{\rho,\rho}, a_{\sigma,\sigma}, a_{\tau,\tau}) = (u_\rho^2, u_\sigma^2, u_\tau^2)$, and moreover $v \rightarrow \alpha_v$ is an isomorphism $A^3 \xrightarrow{\sim} \mathcal{H}^2(G, A)$. What group of order 8 is U in the following cases?

$$v = (1, 1, 1), \quad v = (a, 1, 1), \quad v = (a, a, 1), \quad v = (a, a, a).$$