GROUP COHOMOLOGY HOMEWORK 2

1 The Herbrand quotient and an application. Let the notation be as in the section **10** on finite cyclic groups of my Tuesday Oct.21 handout. So G is cyclic of finite order n, generated by τ. Let $\mathcal{A}$ be the category of G-modules A such that $\hat{\mathcal{H}}^r(G, A)$ is a finite group for all $r \in \mathbb{Z}$. Recall that, by periodicity, it is enough to check finiteness for $\hat{\mathcal{H}}^{-1}(G, A) = \mathrm{Ker}(N)/\mathrm{Im}(\tau - 1)$ and $\hat{\mathcal{H}}^0(G, A) = \mathrm{Ker}(\tau - 1)/\mathrm{Im}(N)$. For $A \in \mathcal{A}$ we define the 'Herbrand quotient'

$$ q(A) = \frac{|\hat{\mathcal{H}}^0(G, A)|}{|\hat{\mathcal{H}}^{-1}(G, A)|} \in \mathbb{Q}^*. $$

For example, $\mathbb{Z} \in \mathcal{A}$ and $q(\mathbb{Z}) = n$. If L/K is cyclic with group G, then L^* is in $\mathcal{A}$ if and only if the 'Norm index' $|(K^*/N_{L/K}L^*|$ is finite, in which case that index is equal to $q(L^*)$. The object of the following string of exercises is to show that the Norm index is equal to the degree for local fields, by using the marvelous properties of the Herbrand quotient. Then to use that fact to derive a nondegenerate pairing $K^*/(K^*)^2 \times K^*/(K^*)^2 \to \{\pm 1\}$ for local K of characteristic not 2, and to compute the pairing explicitly when K has a residue field of odd characteristic, or is the field $\mathbb{Q}_2$ of 2-adic numbers.

Prove the following 11 statements. Of course you may assume statement (n) in proving statement (k) for $k > n$, whether or not you have proved statement (n).

(1) If $0 \to A \to B \to C \to 0$ is an exact sequence of G-modules such that two of the three modules A, B, C are in $\mathcal{A}$, then the third is also, and $q(B) = q(A)q(C)$. [Suggestion: Consider the infinite exact sequence of cohomology associated with the A, B, C sequence. Cupping with the generator $\delta\bar{\chi}$ of $\mathcal{H}^2(G, \mathbb{Z})$ gives an isomorphism of that infinite sequence onto itself raising dimensions by 2. So if we look at 6 successive terms in the infinite sequence , with 5 arrows, the kernel of the first arrow is isomorphic to the cokernel of the 5-th. Now use the fact (which is easily proved by induction on m) that if

$$ 0 \to H_1 \to H_2 \to \cdots \to H_m \to 0 $$

is an exact sequence of finite abelian groups, then the alternating product $\prod_{i=1}^m |M_i|^{(-1)^i}$ of their orders is 1.]

(2) If A is a finite G-module then $q(A) = 1$.

(3) Let K be a field complete with respect to a discrete valuation with finite residue field k. Let L be a cyclic extension of degree n with group $G = <\tau>$. Let A and B be the rings of integers in K and L respectively. Let W be a G-submodule of finite index in B^*. Show that $q(L^*) = nq(W)$.

So we want to find an open submodule W of B^* with $q(W)=1$. To do this we show that for any finite Galois extension L of our field K, cyclic or not, there is a W with $\hat{\mathcal{H}}^r(G, W) = 0$ for all r.

(4) Quoting a theorem from Galois theory, show that there is an $A[G]$-submodule M of L which is isomorphic to $A[G]$ as G-module, and that replacing M by $\pi^i M$ for sufficiently large i, where π is a prime element of A, we can assume also that $M \subset \pi B$ and $M^2 \subset \pi M$.

(5) With the M in (4), put $W_j = 1 + \pi^j M$. Show that the W_j's are G-submodules of finite index in B^*, and that for each $j \geq 0$ the quotient module W_j/W_{j+1} is isomorphic to the group algebra $k[G]$, where $k = A/\pi A$ is the residue field of A.

(6) Let $W = W_0$. Show that $\hat{\mathcal{H}}^r(G, W/W_j) = 0$ for all r and j.

(7) Show that $\hat{\mathcal{H}}^r(G, W) = 0$ for all r.

So now we have proved (see discussion at end of first paragraph) that for finite cyclic extensions L of our local field K, the Norm index $|K^*/N_{L/K}L^*|$ is equal to the degree $[L : K]$

(8) Let K be any field of characteristic $\neq 2$. Define a symbol a, b for a, b in K^* by $(a, b) = 1$ or -1 according to whether a is a Norm from $K(\sqrt{b})$ or not. (If b is a square then interpret this to mean $(a, b) = 1$.) Show that $(a, b) = 1$ if and only if there exist non-zero x, y in K such that $ax^2 + by^2 = 1$. Show also

$$(a, b) = (b, a), \qquad (a, -a) = 1, \qquad (a, 1 - a) = 1,$$

the last assuming $a \neq 1$ as well as $a \neq 0$.

(9) Now suppose again that K is local, as in (3). Show that then the symbol (a, b) is bimultiplicative in the sense that

$$(aa', b) = (a, b)(a', b) \quad \text{and} \quad (a, bb') = (a, b)(a, b').$$

Thus, since our symbol obviously depends on its arguments only up to squares, we have a pairing $K^*/(K^*)^2 \times K^*/(K^*)^2 \to \{\pm 1\}$. Show that this pairing is non-degenerate, meaning that if $(a, b) = 1$ for all a, then b is a square in K.

(10) Suppose K as in (9), and of residue characteristic not 2. Let u and v in A^* be units in K. Let $[u] = 1$ or -1 according as the residue class of u is or is not a square in the residue field (Legendre symbol in unusual notation). Show that the symbol (,) satisfies

$$(u, v) = 1, \qquad (u, \pi) = [u], \qquad (\pi, \pi) = [-1]$$

and note that these rules determine it completely, by symmetry and bimultiplicativity.

(11) Suppose $K = \mathbb{Q}_2$. Show that $|K^*/(K^*)^2| = 8$ and that $R := \{\pm 1, \pm 3, \pm 2, \pm 6\}$ is a set of representatives for the 8 cosets. Make an 8×8 table of the values of (a, b) for $a, b \in R$.