Mathematics 223a Homework due December 4

November 25, 2008

Let L/K be a totally ramified Galois extension of degree p, where K is of finite degree over Q_p, with $A \subset K$ and $B \subset L$ the integral closures of $\mathrm{Z}_p \subset \mathrm{Q}_p$ in their respective fields. Let $\Pi \in B$ be a uniformizer, and $f(X) \in A[X]$ the monic minimal polynomial with Π as a root. Let $\pi := N_{L/K}\Pi \in A$. Let $\mathcal{D}_{L/K} \subset B$ denote the different ideal, and $\mathcal{D}_{L/K}^{-1}$ the co-different fractional ideal in L. We have that $f'(\Pi)$ is a generator of the ideal $\mathcal{D}_{L/K}$. If g is a generator of $\mathrm{Gal}(L/K)$ let $i \geq 0$ be defined by the property that $g(\Pi) = \Pi + \Pi^{i+1}u$ for some appropriate $u \in B^*$. For $n \geq i$ put $\tilde{n} := i + p(n-i)$.

To compute the valuation of the different ideal in this setting, note:

$$f'(\Pi) = \prod_{g \in G; g \neq 1} (\Pi - g(\Pi))$$

and therefore

$$v(f'(\Pi)) = \sum_{g \in G; g \neq 1} v(\Pi - g(\Pi)) = (p-1)(i+1). \quad \text{q.e.d.}$$

Here are some problems to hand in.

Exercise 1 *Let $J \subset L$ be a fractional ideal of L and $I \subset K$ be a fractional ideal of K then*

$$Tr_{L/K}(J) \subset I \quad \Leftrightarrow \quad J \subset I \cdot \mathcal{D}_{L/K}^{-1}.$$

Exercise 2 *For any $n \geq 0$,*

$$Tr_{L/K}(m_L^n) = m_K^{i+1+[\frac{n-i-1}{p}]}.$$

Here [ ] means the greatest integer (i.e., element of Z) $\leq$ what is contained in the brackets.

Exercise 3 *If $x \in m_L^n$ (for $n \geq 0$) then*

$$N_{L/K}(1+x) \equiv 1 + Tr_{L/K}x + N_{L/K}x \mod Tr_{L/K}(m_L^{2n}),$$

Using the exercises above and the fact that $N_{L/K}(m_L^n) = m_K^n$, show that:

Exercise 4 *For $1 \leq n < i$ the norm $N_{L/K}$ induces an isomorphism of groups*

$$B_n^*/B_{n+1}^* \to A_n^*/A_{n+1}^*$$

which is given by $(1+x) \mapsto 1 + N_{L/K}x$ (modulo A_{n+1}^*) and under our "standard identifications"

$$B_n^*/B_{n+1}^* \cong \Pi^n B/\Pi^{n+1}B \cong k \cong \pi^n A/\pi^{n+1}A \cong A_n^*/A_{n+1}^*$$

this isomorphism $k^+ \to k^+$ is given by $\xi \mapsto b \cdot \xi^p$ for some constant $b \in k^*$.

Exercise 5 *For $n > i$ the norm $N_{L/K}$ induces an isomorphism*

$$B_{\tilde{n}}^*/B_{\tilde{n}+1}^* \to A_n^*/A_{n+1}^*$$

which is given by $(1+x) \mapsto 1 + Tr_{L/K}(x)$ (modulo A_{n+1}^*) and under our "standard identifications"this isomorphism $k \to k$ is k-linear, i.e., is given by $\xi \mapsto a \cdot \xi$ for some constant $a \in k^*$.

Exercise 6 *Assume that $i > 0$. For $n = i$ the norm $N_{L/K}$ induces a homomorphism*

$$B_n^*/B_{n+1}^* \to A_n^*/A_{n+1}^*$$

which under the standard identifications yields a homomorphism $k^+ \to k^+$ of the form $\xi \mapsto a \cdot \xi + b \cdot \xi^p$ for constants $a, b \in k^*$.