GROUP COHOMOLOGY LECTURE 3

1. Tensor Product. Let R be a ring, A a right, and B a left R-module. Then $A \otimes_R B$ is defined. One description is as the quotient of $A \otimes_{\mathbb{Z}} B$ by the relations $ar \otimes b = a \otimes rb$ for all $(a, b, r) \in A \times B \times R$.

2. Tor. For fixed A, the functors $B \mapsto \mathrm{Tor}_r^R(A, B)$ are the left derived functors of the right exact functor $B \mapsto A \otimes_R B = \mathrm{Tor}_0^R(A, B)$. They can be computed from a resolution $X_\bigstar$ of A

$$X_\bigstar = \quad \cdots \to X_2 \to X_1 \to X_0 \to A \to 0$$

by projective right R-modules X_r, as

$$\mathrm{Tor}_r^R(A, B) = \mathbb{H}_r(X_\bigstar \otimes_R B).$$

3. Group homology. For a group G and left G-module B, one puts

$$\mathcal{H}_r(G, B) = \mathrm{Tor}_r^{\mathbb{Z}[G]}(\mathbb{Z}, B).$$

We can compute these "homology groups of G with coefficients in B" by the 'standard resolution' in which X_r is the free $\mathbb{Z}$-module with basis elements $\{\sigma_0, \sigma_1, ..., \sigma_r\} \in G^{r+1}$, with G acting on the right by the rule $\{..., \sigma_k, ...\}\tau = \{..., \sigma_k\tau, ...\}$, and the usual boundary maps $\partial : X_r \to X_{r-1}$. Clearly, X_r is a free right $\mathbb{Z}[G]$-module with basis elements

$$[\sigma_1, \sigma_2, ..., \sigma_r] := \{\sigma_1\sigma_2 \cdots \sigma_r,\ \sigma_2 \cdots \sigma_r,\ \ldots, \sigma_{r-1}\sigma_r,\ \sigma_r, 1\}$$

We call elements of $X_r \otimes_{\mathbb{Z}[G]} B$ "standard r-chains of G with coefficients in B". They can be written uniquely in the form

$$\sum_{(\sigma_1, \sigma_2, \ldots, \sigma_r) \in G^r} [\sigma_1, \sigma_2, ..., \sigma_r] \otimes b_{\sigma_1, \sigma_2, \ldots, \sigma_r},$$

where the $b_{\sigma_1, \sigma_2, \ldots, \sigma_r}$ are elements of B, all but a finite number of them 0. We write for simplicity $[\sigma_1, \sigma_2, ..., \sigma_r]b$ instead of $[\sigma_1, \sigma_2, ..., \sigma_r] \otimes b$. The boundary maps are

$$\partial[]b = 0$$

$$\partial[\sigma]b = []\sigma b - []b$$
$$\partial[\sigma, \tau]b = [\sigma]\tau b - [\sigma\tau]b + [\tau]b$$
$$\partial[\rho, \sigma, \tau]b = [\rho, \sigma]\tau b - [\rho, \sigma\tau]b + [\rho\sigma, \tau]b - [\sigma, \tau]b,$$

etc.

4. Low dimensions. We have $\mathcal{H}_0(B,G) = \mathbb{Z} \otimes_{\mathbb{Z}[G]} B = B_G$ for short. Just as $B^G = \mathrm{Hom}_G(\mathbb{Z}, B)$ is the largest submodule of B on which G acts trivially, so $B_G = \mathbb{Z} \otimes_{\mathbb{Z}[G]} B$ is the largest quotient module of B on which G acts trivially. We have $B_G = B/I_G B$, where I_G is the "augmentation ideal" in $\mathbb{Z}[G]$, kernel of the homomorphism $\mathbb{Z}[G] \to \mathbb{Z}$ which maps each σ to 1. This kernel is a free $\mathbb{Z}$-module with basis $\{\sigma - 1\}_{\sigma \in G, \sigma \neq 1}$. If G is generated by elements σ_i, then I_G is the ideal in $\mathbb{Z}[G]$ gemerated by the elements $\sigma_i - 1)$.

The formulas above for ∂ show that $\mathcal{H}_1(G,\mathbb{Z})$ is the quotient of the free $\mathbb{Z}$-module with basis $\{[\sigma]\}_{\sigma \in G}$ by the relations $[\sigma] - [\sigma\tau] + [\tau] = 0$, so we have $\mathcal{H}_1(G,\mathbb{Z}) = G^{\mathrm{ab}} = G/G^c$. More generally, if G acts trivially on B, then $\mathcal{H}_1(G,B) = G^{\mathrm{ab}} \otimes_{\mathbb{Z}} B$, in analogy with $\mathcal{H}^1(G,B) = \mathrm{Hom}_{\mathbb{Z}}(G^{\mathrm{ab}}, B)$.

5. Characterization by properties. The functors $\mathcal{H}_r(G,B)$ for fixed G can be characterized up to unique isomorphism in the analogous way as the $\mathcal{H}^r(G,B)$. In dimension 0 replace A^G by A_G. Require the connecting homomorphisms to lower dimension by 1 instead of raising it by 1, and require $H_r(G, \mathbb{Z}[G] \otimes_{\mathbb{Z}} C) = 0$ instead of $H^r(G, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], C)) = 0$. Dimension shift with the surjection $\mathbb{Z}[G] \otimes_{\mathbb{Z}} B \to B$ instead of with the injection $B \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], B)$.

6. Corestriction, deflation. The homology groups $\mathcal{H}^r(G,B)$ vary covariantly with both G and B. Thus, for $H \subset G$, there are "corestriction" maps cores : $\mathcal{H}_r(H,B) \to \mathcal{H}_r(G,B)$ defined on standard chains by $[..., h_i, ...]b \mapsto [..., h_i, ...]b$, i.e., by viewing an H-chain with coefficients in a G-module B as a G-chain. For example, $\mathcal{H}_1(H,\mathbb{Z}) \to \mathcal{H}_1(G,\mathbb{Z})$ is the homomorphism $H^{\mathrm{ab}} \to G^{\mathrm{ab}}$ induced by the inclusion map $H \hookrightarrow G$.

Similarly, if H is normal in G, then we have "deflation" maps defl : $H_r(G,B) \to \mathcal{H}_r(G/H, B_H)$ obtained by mapping a chain of G in B to the chain of G/H in B_H by replacing the σ's by their cosets σH and viewing the coefficients b mod $I_H B$.

7. G finite, $\mathcal{H}_r = \mathcal{H}^{-1-r}$ if $r > 0$! From now on, assume G is finite. Then we can sum over G and have the operator $N = N_G = \sum_{\sigma \in G} \sigma \in \mathbb{Z}[G]$. For any left G-module A the map $N : A \to A$ induces a map $N^* : A_G \to A^G$, that is, $N^* : \mathcal{H}_0(G,A) \to H^0(G,A)$. Define modified cohomology groups $\hat{\mathcal{H}}^r$ for all $r \in \mathbb{Z}$ as follows:

$$\hat{\mathcal{H}}^r(G,A) = \mathcal{H}^r(G,A), \qquad \text{if} \quad r > 0$$

$$\hat{\mathcal{H}}^0(G,A) = A^G/NA = \mathrm{Coker}\, N^*$$

$$\hat{\mathcal{H}}^{-1}(G,A) = A_{N=1}/IA = \mathrm{Ker}\, N^*$$

$$\hat{\mathcal{H}}^{-1-r}(G,A) = \mathcal{H}_r(G,A), \qquad \text{if} \quad r > 0.$$

For example, $\hat{\mathcal{H}}^{-2}(G,\mathbb{Z}) = \mathcal{H}_1(G,\mathbb{Z}) = G^{\mathrm{ab}}$

For fixed finite G, the functors $A \mapsto \hat{\mathcal{H}}(G,A)$ for $r \in \mathbb{Z}$ are characterized up to unique homomorphism by:

(1)$\hat{\mathcal{H}}^0(G,A) = A^G/N_G A$.

(2) If $0 \to A \to B \to C \to 0$ is exact, then there are connecting homomorphisms $\delta : \hat{\mathcal{H}}^r(G,C) \to \hat{\mathcal{H}}^{r+1}$, which are functorial in the exact sequence, and give an infinite exact sequence extending the usual one for the cohomology groups $\mathcal{H}^r$ for $r > 0$ down to $-\infty$.

(3) $\hat{\mathcal{H}}^r(G,Y) = 0$ for all $r \in \mathbb{Z}$ if $Y = \mathbb{Z}[G] \otimes_{\mathbb{Z}} C \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], C)$, the last arrow being given by $\sum_\sigma \sigma \otimes c_\sigma \mapsto (\sigma \mapsto c_\sigma)$. This is bijective when G is finite, the inverse arrow being $f \mapsto \sum \sigma \otimes f(\sigma)$.

Thus one can dimension shift both up and down with the $\hat{\mathcal{H}}^r$ to prove that these properties do characterize the $\hat{\mathcal{H}}^r$.

To compute the $\hat{\mathcal{H}}^r(G,A)$ one can use an infinite standard cochain complex $C^*(G,A)$ defined by $C^r(G,A)$ being the usual r-cochains $f_r : G^r \to A$ for $r \geq 0$ and $C^{-1-r}(G,A) = C_r(G,A) = X_r \otimes_{\mathbb{Z}[G]} A$ for $r \geq 0$. The coboundary maps $\delta : C^r \to C^{r+1}$ are the usual ones for $r \geq 0$, and are the boundary maps $\partial : C_r \to C_{r-1}$, that is, $C^{-1-r} \to C^{-r}$, for $r > 0$. The new one is the map$[]a \mapsto N_G a$ from $C^{-1} = C_0$ to $C^0 = A$ which connects the chain complex to the cochain complex.

8. Cores and res in all dimensions. The restriction maps $\mathcal{H}^r(G,A) \to \mathcal{H}^r(H,A)$ extend uniquely to all $\hat{\mathcal{H}}^r$ in such a way that they commute with connecting homomorphisms for short exact sequences of G-modules. In terms of standard cochains they are given in negative dimensions by a sum over right cosets $C = H\sigma$. For each such C, let $\overline{C} \in C$ be a chosen reresentative, so that $C = H\overline{C}$. Note that $N_G = \sum_C N_H \overline{C}$. Here and in the following all sums are over the cosets C of H in G. Here are the cochain restriction maps in negatve dimensions:

$$\mathrm{res}[]a = \sum []\overline{C}a$$

$$\mathrm{res}[\sigma]a = \sum [\overline{C}\,\sigma\,\overline{C\sigma}^{-1}]\overline{C\sigma}a$$

$$\mathrm{res}[\sigma,\tau]a = \sum [\overline{C}\,\sigma\,\overline{C\sigma}^{-1}, \overline{C\sigma}\,\tau\,\overline{C\sigma\tau}^{-1}]\overline{C\sigma\tau}a,$$

etc. To prove that these formulas do the job one has only to check that they commute with ∂, that is, $\partial_H \,\mathrm{res}\, x_r = \mathrm{res}\, \partial_G x_r$ for r-chains x and every $r \geq 0$. To find such formulas one starts with $r = 0$, knowing the res map on $(\partial[]a)() = N_{G/H} a$ and guesses the formula for res in dimension 0 which will give the commutativity with ∂. Then continue: Knowing the formula in dimension r, one guesses the one in dimension $r+1$, and soon a pattern appears.

Notice that $\hat{\mathcal{H}}^{-2}(G,\mathbb{Z}) = \mathcal{H}_{-1}(G,\mathbb{Z}) = G^{\mathrm{ab}}$ and that the second formula above shows that the restriction map is the transfer or Verlagerung map $G^{\mathrm{ab}} \to H^{\mathrm{ab}}$ of the homework. As the formulas show, all that is needed to have a restriction map from G to H in group homology is that the index $(G:H)$ be finite.

In that case there is a corestriction map in cohomology mapping $\mathcal{H}^r(H,A)$ to $H^r(G,A)$ which extends the corestriction of homology when G is finite. The explicit formulas for it are sums over left cosets $C = \overline{C}H$. They are the unique maps which commute with connecting homomorphisms and are $a \mapsto \sum \overline{C}a$ from A^H to A^G in dimension 0. They are like the Norm from the fixed field of H to the fixed field of G in the Galois situation, just as restriction is like the inclusion of the latter field in the former. The formula

$$\mathrm{cores} \circ \mathrm{res} = (G : H)$$

holds in all dimensions. In terms of standard cochains, a formula for the corestriction in dimensions > 0 is

$$(\mathrm{cores}\, f_r)(\sigma_1, \sigma_2, ..., \sigma_r) = \sum_C \overline{\tau_1 C} f_r(\overline{\tau_1 C}^{-1} \sigma_1 \overline{\tau_2 C}\,, \overline{\tau_2 C}^{-1} \sigma_2 \overline{\tau_3 C}\,, ..., \overline{\tau_r C}^{-1} \sigma_r \overline{C}),$$

where $\tau_i = \sigma_i \sigma_{i+1} \cdots \sigma_r$ for $1 \le i \le r$, so $\sigma_i \tau_{i+1} = \tau_i$ and the elements $\overline{\tau_i C}^{-1} \sigma_i \overline{\tau_{i+1} C}$ are in H.

9. Semilocal theory. For simplicity, we assume G is finite and we deal with the groups $\hat{\mathcal{H}}^r$. (There are separate theories for $\mathcal{H}^r$ and $\mathcal{H}_r$ when G is infinite, using coinduced and induced modules, respectively, in the two cases.) So let H be a subgroup of a finite group G and let $C = \overline{C}H$ be right cosets so that the C's are permuted by $C \mapsto \sigma C$. A G-module A is 'semilocal for H' or 'induced from H' if there is an H-submodule $B \subset A$ such that A is the direct sum of the subgroups $\overline{C}B$. Then A is determined up to isomorphism by G, H, and B, so the groups $\hat{\mathcal{H}}^r(G, A)$ should be determined by the groups $\hat{\mathcal{H}}^r(H, B)$. In fact, they are equal. More precisely, let p be the projection of A onto B which is identity on B and kills the $\overline{C}B$ for $C \neq H$. Then p is an H-homomorphism and the theorem is that the composed map

$$\hat{\mathcal{H}}^r(G, A) \xrightarrow{\mathrm{res}} \hat{\mathcal{H}}^r(H, A) \xrightarrow{p} \hat{\mathcal{H}}^r(H, B)$$

is a bijection for all $r \in \mathbb{Z}$, the inverse map being the composed map

$$\hat{\mathcal{H}}^r(H, B) \xrightarrow{incl.} \hat{\mathcal{H}}^r(H, A) \xrightarrow{\mathrm{cores}} \hat{\mathcal{H}}^r(G, A).$$

10. Cup products in all dimensions! A bilinear map $A \times B \to C$ of G-modules which is G-bilinear in the sense that $\sigma a \cdot \sigma b = \sigma(a \cdot b)$ induces cup products

$$\mathcal{H}^r(G, A) \times \mathcal{H}^s(G, B) \to \mathcal{H}^{r+s}(G, C)$$

for $r, s \ge 0$ These are characterized by the fact that in dimension 0 the pairing $A^G \times B^G \to C^G$ is induced by the original pairing and by the fact that they commute with connecting homomorphisms when a short exact sequence is paired with a module to another short exact sequence. In terms of standard cochains they are induced by the cochain product

$$(f_r \cup g_s)(\sigma_1, ...\sigma_{r+s}) := f_r(\sigma_1, ..., \sigma_r) \cdot \sigma_1 \cdots \sigma_r g_s(\sigma_{r+1}, ..., \sigma_{r+s})$$

which obeys the rule

$$\delta(f_r \cup g_s) = \delta f_r \cup g_s + (-1)^r f_r \cup \delta g_s.$$

In case G is finite these products extend to pairings $\hat{\mathcal{H}}^r \times \hat{\mathcal{H}}^s \to \hat{\mathcal{H}}^{r+s}$ for all r, s, positive or negative. The standard cochain formulas extend too. For example for $r \geq 0$ and $s < 0$ we have, putting $t = -1 - s \geq 0$,

$$f_r \cup [\sigma_1, ..., \sigma_t]b = [\sigma_1, ..., \sigma_{t-r}](f_r(\sigma_{t-r+1}, ..., \sigma_t) \cdot \sigma_{t-r+1} \cdots \sigma_t b), \quad \text{for} \quad r \leq t,$$

and for $r > t$,

$$(f_r \cup [\tau_1, ..., \tau_t]b)(\sigma_1, ..., \sigma_{r-t-1}) = \sum_{\rho \in G} f_r(\sigma_1, ..., \sigma_{r-t-1}, \rho, \tau_1, ..., t_s) \cdot \sigma_1 \cdots \sigma_{r-t-1} \rho \tau_1, \cdots t_s b.$$

For example, with $r = 2$ and $s = -2$, we have

$$(f_2 \cup [\tau]b)() = \sum_{\rho \in G} f_2(\rho, \tau) \cdot \rho \tau b$$

So the Nakayama map of the homework, and the pairing

$$\mathcal{H}^2(G, A) \times G^{\text{ab}} \to A^G / N_G A$$

it induces, come from the cup product

$$\hat{\mathcal{H}}^2(G, A) \times \hat{\mathcal{H}}^{-2}(G, \mathbb{Z}) \to \hat{\mathcal{H}}^0(G, A)$$

induced by the obvious pairing $A \times \mathbb{Z} \to A$.

The cup product is associative and skew commutative ($f_r \cup g_s = (-1)^{rs} g_s \cup f_r$). The cup product makes $\hat{\mathcal{H}}^*(G, \mathbb{Z})$ into a $\mathbb{Z}$-graded ring for which $\hat{\mathcal{H}}^*(G, A)$ is a graded two-sided $\hat{\mathcal{H}}^*(G, \mathbb{Z})$-module for every A.

10. Finite cyclic G. Suppose $G = \{1, \tau, ..., \tau^{n-1}\}$ is cyclic of order n. I claim then that $\hat{\mathcal{H}}^*(G, \mathbb{Z}) = (\mathbb{Z}/n\mathbb{Z})[x, x^{-1}]$ is the ring of Laurent polynomials with coefficients in $\mathbb{Z}/n\mathbb{Z}$, generated by the class x of the cocycle $\delta\bar{\chi}$ where $\bar{\chi} : G \to \mathbb{Q}$ is a the 1-cochain $\bar{\chi}(\tau^i) = i/n$, for $0 \leq i < n$ representing the homomorphism $\chi : G \to \mathbb{Q}/\mathbb{Z}$ such that $\chi(\tau^i) = i/n \pmod{\mathbb{Z}}$. A trivial calculation shows that $(\delta\bar{\chi} \cup \tau)() = n\bar{\chi}(\tau) = 1$. Hence the class $\tau \in \hat{\mathcal{H}}^{-2}(G, \mathbb{Z})$ is the inverse to x in the ring $\hat{\mathcal{H}}^*(G, \mathbb{Z})$. It follows that, for every A and r, the map $\hat{\mathcal{H}}^r(G, A) \to \hat{\mathcal{H}}^{r+2}(G, A)$ obtained by cupping with x is an isomorphism. For cyclic finite G, the cohomology is periodic with period 2. (But notice that the isomorphisms showing the periodicity do depend on a choice of generator τ of G.) For even r, $\hat{\mathcal{H}}^r(G, A) = \hat{\mathcal{H}}^0(G, A) = \text{Ker}(\tau - 1)/\text{Im}(N_G)$, where the group ring elements $\tau - 1$ and $N_G = 1 + \tau + \tau^2 + \cdots + \tau^{n-1}$ are interpreted as maps $A \to A$, and for odd r, $\hat{\mathcal{H}}^r(G, A) = \hat{\mathcal{H}}^{-1}(G, A) = \text{Ker}(N_G)/\text{Im}(\tau - 1)$, since $\tau - 1$ generates the ideal I_G. Since even for non-cyclic G we have $\hat{\mathcal{H}}^0(G, Z) = \mathbb{Z}/n\mathbb{Z}$ and $\hat{\mathcal{H}}^{-1}(G, Z) = 0$, the ring $\hat{\mathcal{H}}^*(G, \mathbb{Z})$ is indeed the ring of Laurent polynomials as claimed.

11. Herbrand quotient. See **GROUP COHOMOLOGY HOMEWORK 2.**