Adeles

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1 Adeles and Ideles

Now let K be a number field, i.e., of finite degree over $\mathbf{Q}$. Let $\mathcal{O}_K$ be is ring of integers.

Let S be a finite set of primes (i.e., old-fashioned rational primes). Define the (discrete topological) ring $\mathcal{O}_K^S := \mathcal{O}_K[1/N]$ where $N := N(S)$ is the product of the primes in N. So,

$$\text{Spec } \mathcal{O}_K^S = \text{Spec } \mathcal{O}_K \ - \ S,$$

and

$$\text{dir. lim.}_S \ \mathcal{O}_K^S \ = \ K.$$

We will be letting v range through the set of all "primes" of K (synonym: *places* of K) i.e., the nonarchimedean primes corresponding to the maximal ideals of $A \subset K$, its ring of algebraic integers, and the $r_1 + r_2$ archimedean primes[1].

Let K_v denote the completion of K at v. If v is nonarchimedean, let $\mathcal{O}_v := \mathcal{O}_{K_v}$ denote the completion of $\mathcal{O}_K$.

Define the locally compact topological ring,

$$\mathbf{A}_K^S := \prod_{v \text{ arch}} K_v \ \times \ \prod_{p \in S} \{ \prod_{v \ | \ p} K_v \} \ \times \ \prod_{v \ | \ p \notin S} \mathcal{O}_v,$$

or equivalently,

$$\mathbf{A}_K^S := K \otimes_{\mathbf{Q}} \mathbf{R} \ \times \ \prod_{p \in S} K \otimes_{\mathbf{Q}} \mathbf{Q}_p \ \times \ \prod_{p \notin S} \mathcal{O}_K \otimes_{\mathbf{Z}} \mathbf{Z}_p,$$

[1] Everything we say here can be rephrased for fields K that are given as finite separable extensions of $\mathbf{F}_q(t)$, where $\mathcal{O}_K$ is the integral closure of $\mathbf{F}_q[t]$ in K and "archimedean places" has the analogous definition.

noting that $\mathbf{A}_K^S$ is indeed locally compact, and that for $S \subset S'$ an inclusion of finite sets of places as above, we have a natural inclusion

$$\mathbf{A}_K^S \hookrightarrow \mathbf{A}_K^{S'}$$

where the smaller topological ring is imbedded as an open subring of the larger. We have a natural injection of rings

$$\mathcal{O}_K^S \ \hookrightarrow \ \mathbf{A}_K^S$$

allowing us to think of $\mathbf{A}_K^S$ as an $\mathcal{O}_K^S$-algebra. Moreover, everything comes by base change from $\mathbf{Q}$, in the following sense:

$$\mathbf{A}_K^S \ \simeq \ \mathcal{O}_K^S \otimes_{\mathcal{O}_{\mathbf{Q}}^S} \mathbf{A}_{\mathbf{Q}}^S,$$

so we may also think of $\mathbf{A}_K^S$ as an $\mathbf{A}_{\mathbf{Q}}^S$-algebra. As $\mathbf{A}_{\mathbf{Q}}^S$-module one checks that $\mathbf{A}_K^S$ is free of rank $[K:\mathbf{Q}]$, and we have the relative version of this statement for any finite extension of number fields L/K; namely, $\mathbf{A}_L^S$ is an $\mathbf{A}_K^S$-algebra that is finite flat (and free of rank $[L:K]$ as $\mathbf{A}_K^S$-module)—and, moreover—is obtained by base change:

$$\mathbf{A}_L^S \ \simeq \ \mathcal{O}_L^S \otimes_{\mathcal{O}_K^S} \mathbf{A}_K^S,$$

Define $\mathbf{A}_K$, **the ring of adeles of** K, to be:

$$\mathbf{A}_K := \bigcup_S \mathbf{A}_K^S = \lim_S \mathbf{A}_K^S.$$

This structure is compatible with augmentation of S so we also have the "same" structure in the limit, i.e.:

$$K \ \hookrightarrow \ \mathbf{A}_K,$$

allowing us to think of $\mathbf{A}_K$ as a K-algebra, and for any finite extension of number fields L/K; namely, $\mathbf{A}_L$ is an $\mathbf{A}_K$-algebra that is finite flat (and free of rank $[L:K]$ as $\mathbf{A}_K$-module) obtained, again, by base change:

$$\mathbf{A}_L \ \simeq \ L \otimes_K \mathbf{A}_K.$$

The image of K in $\mathbf{A}_K$ is called the subfield of **principal adeles**.

Since $\mathbf{A}_K$ is a locally compact topological group it has a Haar measure $\mu = \mu_{\mathbf{A}_K}$, unique up to scalar normalization, and this measure can be taken to be the (naive"restricted" product measure). That is, we view $\mathbf{A}_K^S$ as a product of local factors and give it the product of local Haar measures μ_v on K_v (for all v) and on $\mathcal{O}_v$ if v is nonarchimedean) such that the integral of μ_v over the unit disc in K_v is equal to 1. The natural injections $\mathbf{A}_K^S \hookrightarrow \mathbf{A}_K^{S'}$ are measure preserving, so induce a measure $\mu_{\mathbf{A}_K}$ on the union. There are more sophisticated choices of normalization of Haar measure on $\mathbf{A}_K$, but this will do for a while.

At this point in our discussion, our level of knowledge about the topological nature of $\mathbf{A}_K$ as K-algebra, and as $\mathbf{A}_{\mathbf{Q}}$-module, is somewhat different. For the first, we have no idea yet what topological properties the injection $K \hookrightarrow \mathbf{A}_K$ enjoys. But for the second, it is quite straightforward to see that giving K a $\mathbf{Q}$-basis $\{\alpha_1, \alpha_2, \ldots, \alpha_d\}$ with $d = [K : \mathbf{Q}]$, we may write $\mathbf{A}_K$ as

$$\mathbf{A}_K \ = \ \alpha_1 \cdot \mathbf{A}_{\mathbf{Q}} \ \oplus \ \alpha_2 \cdot \mathbf{A}_{\mathbf{Q}} \ \oplus \ \ldots \ \oplus \ \alpha_d \cdot \mathbf{A}_{\mathbf{Q}}.$$

This $\mathbf{A}_K$ has a basis of $[K : \mathbf{Q}]$ elements rendering it isomorphic to the product topological $\mathbf{A}_{\mathbf{Q}}$-module,

$$\mathbf{A}_{\mathbf{Q}} \times \mathbf{A}_{\mathbf{Q}} \times \ldots \times \mathbf{A}_{\mathbf{Q}}$$

(with $[K : \mathbf{Q}]$ factors).

Theorem 1 *The imbedding $K \hookrightarrow \mathbf{A}_K$ identifies K with a discrete subring in $\mathbf{A}_K$, and the quotient (additive topological group) $\mathbf{A}_K/K$ is compact.*

Proof: The above discussion allows us to reduce to the case $K = \mathbf{Q}$. Taking the *open box*

$$\{\alpha \in \mathbf{A}_{\mathbf{Q}} \mid |\alpha_\infty|_\infty < 1/2; \text{ and } |\alpha_p|_p \leq 1 \ \ for \ \ p < \infty,\}$$

we immediately see that 0 is the only principal adele in that box, giving (by translation) discreteness of the image of $\mathbf{Q}$ in $\mathbf{A}_{\mathbf{Q}}$, and hence of K in $\mathbf{A}_K$ for any number field. Taking the compact

$$D := \{\alpha \in \mathbf{A}_{\mathbf{Q}} \ |\alpha_p|_p \leq 1 \ \ for \ \ p \leq \infty,\}$$

we show that it is a *sloppy fundamental domain* in that

$$\mathbf{Q} \cdot \mathbf{D} = \mathbf{A}_{\mathbf{Q}}$$

(by an application of the chinese remainder theorem). This gives that $\mathbf{A}_{\mathbf{Q}}/\mathbf{Q}$ is compact, and hence so is $\mathbf{A}_K/K$ for any number field. These quotients also have finite volume.

Let $c = (\ldots, c_v, \ldots) \in \mathbf{A}_K$ with almost all c_v equal to 1, and no c_v equal to 0. Denote its norm by norm $C := \prod_v |c_v|_v \neq 0$ and consider the box

$$D(c) == \{\alpha \in \mathbf{A}_K \ |\alpha_v|_v \leq |c_v|_v, \text{for all } v\}$$

so that the volume of $D(c)$ is C.

Theorem 2 *There is a finite C such that if the box $D(c)$ has volume $\geq C$, then $D(c)$ contains some nonzero principal adele.*

Proof: Let e be the adele $e_v = c_v/2$ for v archimedean and $e_v = c_v$ for v non archimedean so that the volume of $D(e)$ is $C_o := C/2^{[K:\mathbf{Q}]}$ Take $C_o > vol(\mathbf{A}_K/K)$ implying the existence of at least two elements of $delta_1, \delta_2 \in D(c)$ differing by a nonzero principal adele α, so $|\alpha_v|_v < |c_v|_v$ for v nonarchimedean and real. We then take any $C > 2^{[K:\mathbf{Q}]} vol(\mathbf{A}_K/K)$.