The Formulas

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1 The integral W

Let r_1, r_2 be non-negative integers and put $n = n(r_1, r_2) = r_1 + 2r_2$. Let $a > 0$ and define

$$W_{r_1,r_2}(a) := \int u_{r_1+1} u_{r_1+2} \cdots u_{r_1+r_2} du_1 du_2 \cdots du_{r_1+r_2}$$

where the integral is taken over the region defined by the inequalities $u_i \geq 0$ and $\sum_{i=1}^{i=r_1+r_2} u_i \leq a$.

Compute this in three steps:

1. $W_{r_1,r_2}(a) = a^n \cdot W_{r_1,r_2}(1)$ (where $n = n(r_1, r_2)$).

2. If $r_1 > 0$, then

$$W_{r_1,r_2}(1) = \int_{u_{r_1}=0}^{u_{r_1}=1} W_{r_1-1,r_2}(1-u_{r_1}) du_{r_1}$$

$$= W_{r_1-1,r_2}(1) \int_{u_{r_1}=0}^{u_{r_1}=1} (1-u_{r_1})^{n-1} du_{r_1}$$

$$= \frac{1}{n} W_{r_1-1,r_2}(1) \;=\; \frac{1}{n(n-1)} W_{r_1-2,r_2}(1) = \ldots$$

$$= \frac{1}{n(n-1)\cdots(n-r_2+1)} W_{0,r_2}(1).$$

3. (a)

$$W_{0,r_2}(1) \;=\; \int_{u_{r_2}=0}^{u_{r_2}=1} W_{0,r_2-1}(1-u_{r_2}) u_{r_2} du_{r_2}$$

$$=\; W_{0,r_2-1}(1) \int_{u_{r_2}=0}^{u_{r_2}=1} (1-u_{r_2})^{2r_2-2} u_{r_2} du_{r_2}$$

(b) and since (change of variables $1-u=v$)

$$\int_{u=0}^{u=1}(1-u)^{n-2}udu \;=\; \int_{v=0}^{v=1}v^{n-2}(1-v)dv$$
$$= \int_{v=0}^{v=1}v^{n-2}dv - \int_{v=0}^{v=1}v^{n-1}dv \;=\; \frac{1}{n(n-1)}$$

we get

(c)

$$W_{0,r_2}(1) \;=\; \frac{1}{2r_2!}$$

and therefore:

4. (a)

$$W_{r_1,r_2}(1) \;=\; \frac{1}{n!}$$

(b)

$$W_{r_1,r_2}(a) \;=\; \frac{a^n}{n!}$$

2 The volume of $\mathcal{E}(a)$

Here we are working in $\mathbf{R}^{r_1}\times\mathbf{C}^{r_2} \;\simeq\; \mathbf{R}^n$ and define

$$\mathcal{E}(a) := \{(x_i,z_j)\in\mathbf{R}^{r_1}\times\mathbf{C}^{r_2} \;\mid\; \sum_{i=1}^{i=r_1}|x_i| \;+\; 2\sum_{j=1}^{j=r_2}|z_j|\le a\}.$$

Then

1. if we put

$$\mathcal{E}^+(a) := \{(x_i,z_j)\in\mathcal{E}(a)\mid x_i\ge 0(i=1,2,\ldots,r_1)\}$$

we have:

$$\mathrm{vol}(\mathcal{E}(a)) \;=\; 2^{r_1}\cdot\mathrm{vol}(\mathcal{E}^+(a))$$

2. and if we make the change of variables $\{x_i;z_j\}$ to $\{u_1,u_2,\ldots,u_{r_1+r_2};\theta_{r_1+1},\theta_{r_1+2},\ldots,\theta_{r_1+r_2}\}$ where $u_i=x_i$ for $i=1,2,\ldots,r_1$ and $2z_j=u_{r_1+j}\theta_{r_1+j}$ for $j=1,2,\ldots,r_2$, we get that

$$\mathrm{vol}(\mathcal{E}(a)) \;=\; 2^{r_1}4^{-r_2}(2\pi)^{r_2}W_{r_1,r_2}(a)$$

and so:

$$\mathrm{vol}(\mathcal{E}(a)) \;=\; 2^{r_1}4^{-r_2}(2\pi)^{r_2}\frac{a^n}{n!}.$$

3 Catching integral points of small "funny norm"

Let $K, \mathcal{O}_K, d_K$ be what you think they are[1], with $n = r_1 + 2r_2$ and let $I \subset \mathcal{O}_K$ be a nonzero ideal, so that the volume of the lattice $I \subset \mathcal{O}_K \subset \mathbf{R}^{r_1} \times \mathbf{C}^{r_2}$ is

$$\mathrm{vol}(I) \;=\; 2^{-r_2} N I d_K^{\frac{1}{2}}.$$

Then by Blichfeldt we have a lattice point $\alpha \in I$ such that

$$\sum_{i=1}^{n} |\sigma_i(\alpha)| \leq a$$

if the volume of $\mathcal{E}(a)$ is $\geq 2^n \cdot \mathrm{vol}(I)$.

Defining, then, the number a by the equality:

$$\mathrm{vol}(\mathcal{E}(a)) = 2^n \cdot \mathrm{vol}(I)$$

we get:

$$a^n := n! \cdot (\frac{4}{\pi})^{r_2} \cdot NI \cdot d_K^{\frac{1}{2}}$$

as a defining equation for a.

Since

$$|N_K^{\mathbf{Q}}(\alpha)|^{\frac{1}{n}} \leq \frac{1}{n} \sum_{i=1}^{n} |\sigma_i(\alpha)| \leq \frac{a}{n},$$

we have that:

$$|N_K^{\mathbf{Q}}(\alpha)| \leq \frac{1}{n} \sum_{i=1}^{n} |\sigma_i(\alpha)| \leq \frac{a^n}{n^n} = \frac{n!}{n^n} (\frac{4}{\pi})^{r_2} N I d_K^{\frac{1}{2}}$$

and therefore

Theorem 1 *1. There is a point $\alpha \in I$ of norm*

$$\leq \frac{n!}{n^n} (\frac{4}{\pi})^{r_2} N I d_K^{\frac{1}{2}}.$$

2. Any ideal class of K has an ideal of norm

$$\leq \frac{n!}{n^n} (\frac{4}{\pi})^{r_2} d_K^{\frac{1}{2}}.$$

3. The absolute value of the discriminant of K is

$$\geq \frac{n^{2n}}{(n!)^2} (\frac{\pi}{4})^{2r_2}.$$

[1] $d_K := |\mathrm{disc}_K|$