NOTES FOR COHOMOLOGY LECTURE 4

4.1. Finite cyclic G. Suppose $G = \{1, \tau, ..., \tau^{n-1}\}$ is cyclic of order n. I claim then that $\hat{\mathcal{H}}^*(G,\mathbb{Z}) := \oplus_{r\in\mathbb{Z}}\hat{\mathcal{H}}^r(G,\mathbb{Z}) = (\mathbb{Z}/n\mathbb{Z})[x, x^{-1}]$ is the ring of Laurent polynomials with coefficients in $\mathbb{Z}/n\mathbb{Z}$, generated by the class $x \in \mathcal{H}^2(G,\mathbb{Z})$ of the 2-cocycle $\delta\bar{\chi}$ where $\bar{\chi} : G \to \mathbb{Q}$ is a the 1-cochain $\bar{\chi}(\tau^i) = i/n$, for $0 \le i < n$ representing the homomorphism $\chi : G \to \mathbb{Q}/\mathbb{Z}$ such that $\chi(\tau^i) = i/n \pmod{\mathbb{Z}}$. A trivial calculation shows that $(\delta\bar{\chi} \cup [\tau])() = n\bar{\chi}(\tau) = 1$. Hence the class of $[\tau]$, in $\hat{\mathcal{H}}^{-2}(G,\mathbb{Z}) = G$, is the inverse to x in the ring $\hat{\mathcal{H}}^*(G,\mathbb{Z})$. It follows that, for every A and r, the map $\hat{\mathcal{H}}^r(G,A) \to \hat{\mathcal{H}}^{r+2}(G,A)$ obtained by cupping with x is an isomorphism. For finite cyclic G, the $\hat{\mathcal{H}}$ cohomology is periodic with period 2. (But notice that the isomorphisms showing the periodicity do depend on a choice of generator τ of G.) For even r, $\hat{\mathcal{H}}^r(G,A) = \hat{\mathcal{H}}^0(G,A) = \operatorname{Ker}(\tau-1)/\operatorname{Im}(N_G)$, where the group ring elements $\tau - 1$ and $N_G = 1 + \tau + \tau^2 + \cdots + \tau^{n-1}$ are interpreted as maps $A \to A$, and for odd r, $\hat{\mathcal{H}}^r(G,A) = \hat{\mathcal{H}}^{-1}(G,A) = \operatorname{Ker}(N_G)/\operatorname{Im}(\tau-1)$, since $\tau - 1$ generates the ideal I_G. Since $\hat{\mathcal{H}}^0(G,Z) = \mathbb{Z}/n\mathbb{Z}$ and $\hat{\mathcal{H}}^{-1}(G,Z) = 0$, the ring $\hat{\mathcal{H}}^*(G,\mathbb{Z})$ is indeed the ring of Laurent polynomials as claimed.

4.2 Cohomological triviality and the sur-bi-in jection theorem For a finite group G, a G-module A and an integer k, let $S_k = S_k(G,A)$ denote the property that $\hat{\mathcal{H}}^k(H,A) = 0$ for all subgroups H of G. If $S_k(G,A)$ holds for all $k \in \mathbb{Z}$ we say that A is a cohomologically trivial G-module.

Theorem. *If S_k and S_{k+1} hold for some k then A is cohomologically trivial .*

Lemma. *If $H \triangleleft G$, there are exact sequences*

$$0 \to \hat{\mathcal{H}}^1(G/H, A^H) \to \hat{\mathcal{H}}^1(G,A) \to \hat{\mathcal{H}}^1(H,A) \tag{1}$$

$$0 \leftarrow \hat{\mathcal{H}}^0(G/H, A^H) \leftarrow \hat{\mathcal{H}}^0(G,A) \leftarrow \hat{\mathcal{H}}^0(H,A) \tag{2}$$

$$0 \to \hat{\mathcal{H}}^{-1}(G/H, A_H) \to \hat{\mathcal{H}}^{-1}(G,A) \to \hat{\mathcal{H}}^{-1}(H,A) \tag{3}$$

$$0 \leftarrow \hat{\mathcal{H}}^{-2}(G/H, A_H) \leftarrow \hat{\mathcal{H}}^{-2}(G,A) \leftarrow \hat{\mathcal{H}}^{-2}(H,A). \tag{4}$$

Proof of Lemma: Sequence (1) is the well-known inflation-restriction sequence which is exact in dimension 1.

Sequence (4) is the corestriction-deflation counterpart of (1) for homology. (Recall $\hat{\mathcal{H}}^{-r} = \mathcal{H}_{r-1}$ for $r \ge 2$.)

The non trivial maps in (2) are induced by the identity map $A^G \leftarrow A^G$ and the norm map $N_{G/H} : A^G \leftarrow A^H$, and exactness is easy to check.

(3) is the counterpart of (2) in homology. (Recall $\hat{\mathcal{H}}^{-1} = \hat{\mathcal{H}}_0$.)

Proof of Theorem: We use induction on the order of G. There is nothing to prove if $|G| = 1$. Suppose $|G| > 1$, and the theorem is true for all proper subgpoups of G. If $|G|$ is not a prime power, then, by the induction hypothesis, the theorem is true for all Sylow subgroups of G, and we use the fact that the restriction from G to a p-Sylow subgroup is injective on the p-primary parts of the cohomology groups of G (because cores $\circ$ res is multiplication by the index, which is prime to p). If $|G|$ is a prime power we have to work a bit harder. By induction on k, up and down, it suffices to prove, for all $k \in \mathbb{Z}$, that

(*) $$(*) \qquad S_k \text{ and } S_{k+1} \Rightarrow S_{k+2} \text{ and } S_{k-1}.$$

By 'dimension shifting', we know, for finite G, that for each G-module A and each integer m, there exists a G-module $A^{(m)}$, such that $\hat{\mathcal{H}}^r(H, A) \simeq \hat{\mathcal{H}}^{r-m}(H, A^{(m)})$ for all $r \in \mathbb{Z}$ and all subgroups $H \subset G$. Hence it suffices to prove (*) for one single k. We show that the lemma implies (*) for $k = 0$. Since $|G|$ is a prime power, G has a normal subgroup H such that G/H is cyclic of prime order. The lemma with that H, the periodicity of the cohomology of G/H, and the induction hypothesis proves (*) for $k = 0$.

Corollary 1. *Suppose $f : A \to B$ is an injective G-homomorphism and k is an integer such that the induced maps $f_* : \hat{\mathcal{H}}^r(H, A) \to \hat{\mathcal{H}}^r(H, B)$ are surjective for $r = k$, bijective for $r = k+1$, and injective for $r = k+2$, for all subgroups H of G. Then f_* is bijective for all $r \in \mathbb{Z}$ and all subgroups H of G.*

Proof: Let C be the cokernel of f, and consider the infinite exact sequence of cohomology

$$\cdots \to \hat{\mathcal{H}}^k(H, A) \to \hat{\mathcal{H}}^k(H, B) \to \hat{\mathcal{H}}^k(H, C)$$

$$\to \hat{\mathcal{H}}^{k+1}(H, A) \to \hat{\mathcal{H}}^{k+1}(H, B) \to \hat{\mathcal{H}}^{k+1}(H, C) \to$$

$$\hat{\mathcal{H}}^{k+2}(H, A) \to \hat{\mathcal{H}}^{k+2}(H, B) \to \hat{\mathcal{H}}^{k+2}(H, C) \to \cdots .$$

By exactness, the hypotheses are equivalent with the vanishing of $\hat{\mathcal{H}}^k(H, C)$ and $\hat{\mathcal{H}}^{k+1}(H, C)$. Hence, by the theorem, $\hat{\mathcal{H}}^r(H, C) = 0$ for all $r \in \mathbb{Z}$.

Corollary 2. *Corollary 1 holds for an arbitrary $f : A \to B$. It need not be injective.*

Proof: Let $i : A \hookrightarrow A^*$ be an embedding of A into a cohomologically trivial module A^*, and apply corollary 1 to the injective map $(f, i) : A \to B \times A^*$.

Corollary 3 *Let $\alpha \in \hat{\mathcal{H}}^t(G, A)$. For each integer r, and each subgroup H of G, let $\alpha_H^{(r)} : \hat{\mathcal{H}}^r(H, \mathbb{Z}) \to \hat{\mathcal{H}}^{r+t}(H, A)$ denote the map $\zeta \mapsto (\mathrm{res}_H^G \alpha) \cup \zeta$. If the maps $\alpha_H^{(r)}$ are surjective, bijective and injective respectively, for all H, in three successive dimensions, $r = k-1, k, k+1$, then they are bijective for all r and H.*

Proof: Let $A^{(t)}$ be the dimension shifting module such that $\hat{\mathcal{H}}^s(H,A)$ is canonically isomorphic to $\hat{\mathcal{H}}^{s-t}(H,A^{(t)})$. The canonical isomorphism is a series of connecting homomorphisms if $t<0$, or their inverses if $t>0$. Since cup products commute with connecting homomorphisms, the theorem about A and cupping with $\alpha\in\hat{\mathcal{H}}^t(G,A)$ is equivalent to the theorem about $A^{(t)}$ and cupping with the class $\beta\in\hat{\mathcal{H}}^0(G,A^{(t)})$ corresponding to α. But cupping with β is just the cohomology map induced by an $f\in \mathrm{Hom}_G(\mathbb{Z},A^{(t)})=(A^{(t)})^G$ representing β, so Corollary 3 follows from Corollary 2.

For a version of Corollary 3 for cup products for more general pairings $A\times B\to C$ rather than $A\times\mathbb{Z}\to A$, see Serre, Corps Locaux, Ch.IX,§8, Th.13.

4.3 Formations. A *formation* $(G,\{G_F\}_{F\in\mathcal{F}},A)$ consists of a profinite group G with an indexing of the set of open subgroups of G by a set $\mathcal{F}$, and a discrete G-module A. We denote elements of $\mathcal{F}$ by symbols like $K,L,M....$ and call them 'fields'. The open subgroup corresponding to a field $F\in\mathcal{F}$ is denoted by G_F. Example: F a field, Ω its separable closure, $G=\mathrm{Gal}(\Omega/F)$, $\mathcal{F}=$ the set of finite extension fields of F in Ω, $G_K=\mathrm{Gal}(\Omega/K)$ for $K\in\mathcal{F}$.

We write $K\subset L$ if $G_L\subset G_K$, and indicate this situation symbolically by L/K. We call L/K a *layer* of the formation and say that L is an extension field of K. The *degree* of a layer L/K is denoted and defined by $[L:K]:=(G_K:G_L)$. A layer L/K is *normal* if $G_L\triangleleft G_K$. The *Galois group* of a normal layer L/K is $G_{L/K}:=G_K/G_L$.

For each field K we write $A_K:=A^{G_K}=\mathcal{H}^0(G_K,A)$ and call A_K a *level* of the formation. In the example above, in which $A=\Omega^*$, we have $A_K=K^*$ for each field K. If L/K is a layer, we say A_L is the top level and A_K the bottom level of the layer and have maps, inclusion $=\mathrm{res}^K_L$: $A_K\to A_L$, and norm, $N_{L/K}=\mathrm{cores}^L_K:A_L\to A_K$ between the top and bottom levels in both directions. If L/K is normal, then $G_{L/K}$ acts on the top level A_L, and the bottom level A_K is $A_L^{G_{L/K}}=(A^{G_L})^{G_K/G_L}=\mathcal{H}^0(G_{L/K},A_L)$.

Notation: For each normal layer L/K we put

$$\hat{\mathcal{H}}^r(L/K):=\hat{\mathcal{H}}^r(G_{L/K},A_L),\qquad \text{for } r\in\mathbb{Z},\quad\text{and}$$

$$\mathcal{H}^r(L/K):=\mathcal{H}^r(G_{L/K},A_L),\quad\text{and}\quad \mathcal{H}^r(*/K):=\mathcal{H}^r(G_K,A)\quad\text{for } r\geq 0,$$

In writing $\hat{\mathcal{H}}^r(L/K)$ we imply that L/K is a normal layer.

If $K\subset L\subset M$ we have maps:

$$\mathrm{infl}:\mathcal{H}^r(L/K)\to\mathcal{H}^r(M/K),\quad\text{for } r\geq 0$$

$$\mathrm{res}:\hat{\mathcal{H}}^r(M/K)\to\hat{\mathcal{H}}^r(M/L),\quad\text{for all } r$$

$$\mathrm{cores}:\hat{\mathcal{H}}^r(M/L)\to\hat{\mathcal{H}}^r(M/K),\quad\text{for all } r$$

$$\sigma:\hat{\mathcal{H}}^r(L/K)\to\hat{\mathcal{H}}^r(\sigma L/\sigma K),\quad\text{for all } \sigma\in G,$$

where σM is defined by $G_{\sigma M} = \sigma G_M \sigma^{-1}$. Note that then $A_{\sigma M} = \sigma A_M$. Note also that $\mathcal{H}^r(*/K) = \varinjlim_L \mathcal{H}^r(L/K)$ for $r \geq 0$, relative to the inflation maps

4.4 Class Formations. A *class formation* is a formation $(G, \{G_F\}_{F\in\mathcal{F}}, A)$ satisfying the following conditions:

(CF1) $\hat{\mathcal{H}}^1(L/K) = 0$, for every normal layers L/K.

(CF2) $\hat{\mathcal{H}}^2(L/K)$ is cyclic, of order $[L:K]$, with a canonical generator $\alpha_{L/K}$ for every normal L/K.

(CF3) For $K \subset L \subset M$ with L/K and M/K normal we have $\mathrm{infl}_{M/K}^{L/K} \circ \alpha_{L/K} = [M:L]\alpha_{M/K}$.

(CF4) If $K \subset L \subset M$, with M/K normal, then $res_L^K \alpha_{M/K} = \alpha_{M/L}$.

Note that (CF1) is equivalent to

(CF5) $\mathcal{H}^1(*/K) = 0$ for every field K.

If (CF1) is satisfied, then the inflation-restriction sequence is exact for $r = 2$, and it is an easy exercise to check that (CF2), (CF3) and (CF4) together are equivalent to

(CF6) There is a collection of injective homomorphisms $\mathrm{inv}_K : \mathcal{H}^2(*/K) \to \mathbb{Q}/\mathbb{Z}$, one for each field K, such that $\mathrm{inv}_L \circ \mathrm{res}_L^K = [L:K]\,\mathrm{inv}_K$ for all layers L/K.

(The connection is simply $\mathrm{inv}_{L/K}(\alpha_{L/K}) = \frac{1}{[L:K]}$, where $\mathrm{inv}_{L/K} : \mathcal{H}^2(L/K) \to \mathbb{Q}/\mathbb{Z}$ is the composed map $\mathrm{inv}_K \circ \mathrm{infl}_{*/K}^{L/K}$, i.e., the restriction of inv to $\mathcal{H}^2(L/K)$ if we view the inflation maps as inclusions.)

The class $\alpha_{L/K}$ is called the *fundamental class* of the normal layer L/K.

Theorem. In a class formation the cup product with $\alpha_{L/K}$ gives isomorphisms $\alpha_{L/K}^{(r)} : \hat{\mathcal{H}}^r(G_{L/K}, \mathbb{Z}) \xrightarrow{\sim} \hat{\mathcal{H}}^{r+2}(L/K)$ for all $r \in \mathbb{Z}$ and every normal layer L/K. These isomorphisms commute with restriction, corestriction and conjugation in all dimensions. For $K \subset L \subset M$ with L/K and M/K normal we have $\mathrm{infl}_{M/K}^{L/K} \circ \alpha_{L/K}^{(r)} = [M:L]\alpha_{M/K}^{(r)} \circ \mathrm{infl}_{M/K}^{L/K}$ for all $r \geq 0$, on the groups $\mathcal{H}$.

Proof: That the maps $\alpha^{(r)}$ are isomorphisms follows directly from Corollary 3 above: $\alpha^{(-1)}$ is surjective by (CF1), $\alpha^{(0)}$ is bijective by (CF2) because $\hat{\mathcal{H}}^0(G_{L/K}, \mathbb{Z}) = \mathbb{Z}/[L:K]\mathbb{Z}$, and $\alpha^{(1)}$ is injective automatically, because $\hat{\mathcal{H}}^1(G_{L/K}, \mathbb{Z}) = 0$. The maps $\alpha^{(r)}$ commute with restriction and corestriction by (CF4) and the basic relations between cup products and res and cores. By (CF3), we have for $r \geq 0$

$$(\mathrm{infl} \circ \alpha_{L/K}^{(r)})(\zeta) = \mathrm{infl}(\alpha_{L/K} \cup \zeta) = \mathrm{infl}\, \alpha_{L/K} \cup \mathrm{infl}\, \zeta = [M:L]\alpha_{M/K} \cup \mathrm{infl}(\zeta) = [M:L](\alpha^{(r)} \circ \mathrm{infl})(\zeta).$$