# 1 Zeta Functions of a Number Field

Let $K$ be a field of degree $n$ over $\mathbb{Q}$.

## 1.1 The Dedekind Zeta Function

$$\zeta_K(s) \;=\; \prod_P (1 - \frac{1}{NP^s})^{-1}$$

the product being over all nonzero prime ideals in $\mathcal{O}_K$.

$$\zeta_K(s) \;=\; \sum_{0 \neq I \subset \mathcal{O}_K} \frac{1}{NI^s} \;=\; \sum_{n=1}^{\infty} \frac{a_K(n)}{n^s},$$

where $a_K(n)$ is the number of ideals of $\mathcal{O}_K$ of norm $n$. So, with the notation of last week,

$$M_K(X) = \sum_{n \leq X} a_K(n).$$

We will be dicing this Dirichlet series in various ways. First, we separate ideals into ideal classes. For $\eta \in H_K$,

$$\zeta_K^{\eta}(s) \;=\; \sum_{0 \neq I \subset \mathcal{O}_K; I \in \eta} \frac{1}{NI^s} \;=\; \sum_{n=1}^{\infty} \frac{a_K^{\eta}(n)}{n^s},$$

so $a_K^{\eta}(n)$ is the number of ideals of $\mathcal{O}_K$ in the ideal class $\eta$ that are of norm $n$, and

$$M_K^{\eta}(X) = \sum_{n \leq X} a\eta_K(n),$$

in the notation of last week.

Of course,

$$\zeta_K(s) = \sum_{\eta \in H_K} \zeta_K^{\eta}(s).$$

## 1.2 Residues

**Lemma 1** *If $n \mapsto a(n)$ is any complex-valued function on natural numbers, and if the summation function*

$$M(X) = \sum_{n \leq X} a(n)$$

*satisfies an inequality of the type*

$$|M(X) - \rho \cdot X| < \text{Constant} \cdot X^{\sigma}$$

*for some complex number $\rho$, and and real number $\sigma$ with $0 \le \sigma < 1$ for $X \ge 1$ then the Dirichlet series*

$$D(s) \;=\; \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

*converges in the half-plane $\mathcal{R}(s) > 1$ and $D(s) - \frac{\rho}{s-1}$ has an analytic continuation to the half-plane $\mathcal{R}(s) > \sigma$.*

A consequence of our previous work is the following:

**Corollary 1** *The Dirichlet series $\zeta_K^{\eta}(s)$ has a meromorphic continuation to the half-plane $\mathcal{R}(s) > 1 - 1/n$, and in this region has only one singularity: a simple pole at $s = 1$ with residue equal to*

$$B_K = \frac{2^{r_1}(2\pi)^{r_2} R_K}{|\mu(K)| d_K^{\frac{1}{2}}}.$$

*The Dedekind zeta function has a similar meromorphic continuation, but with residue equal to $B_K |H_K|$.*

## 1.3 Levels

By a **level** let's mean a pair

$$c := (c_f, c_\infty)$$

where $c_f$ is a nonzero ideal in $\mathcal{O}_K$ and $c_\infty$ is a set of real places of $K$. We will say that $c > c'$ if $c_f \subset c_f'$ and $c_\infty' \subset c_\infty$. By "level one" let us mean level $c$ where $c_f = \mathcal{O}_K$ and $c_\infty$ is the empty set.

Form the groups

- $\mathcal{I}_c :=$ the group of fractional ideals $I$ of $K$ that are relatively prime to $c_f$ (in the sense that if $v$ is a nonarchimedean place of $K$ such that $v(c_f) \ne 0$ then $v(I) = 0$).

- $\mathcal{P}_c :=$ the group of principal fractional ideals $I$ of $K$ that can be generated by a field element $a \in K$ such that $a \equiv 1$ modulo $c_f$ and $a_v > 0$ if $v \in c_\infty$.

- $H_{K,c} := \mathcal{I}_c / \mathcal{P}_c$ the **level $c$ ideal class group of** $K$. (So the classical ideal class group of $K$ is, in this terminology, the "level one" ideal class group of $K$.)

If $c < c'$ we have natural surjective homomorphisms

$$J_K/K^* \longrightarrow H_{K,c'} \longrightarrow H_{K,c} \longrightarrow H_K.$$

Here is how obtain these homomorphisms. Let $c$ be any level. Note first, that if $\alpha \in J_K$ is an idele, there is a principal idele (i.e., a nonzero field element) $a \in K^*$ such that

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- $\alpha_v \equiv a$ modulo $1 + c_f \mathcal{O}_v$ in $K_v^*$ if $v$ is nonarchimedean,

- $\alpha_v/a > 0$ if $v \in c_\infty$.

Note second, that if—for a given idele $\alpha$— two field elements $a$ and $a'$ do the above trick, then the ratio $a'/a$ lies in $\mathcal{P}_c$. The mapping $J_K/K^* \to H_{K,c}$ is defined as follows: for $\alpha \in J_K$ find an $a$ as above, and send $\alpha$ to the fractional ideal $\prod_v P_v^{\mathrm{ord}_v(\alpha_v)}/a$, the product being over all nonarchimedean places. Dependence on the choice of $a$ is erased after passing to the quotient by $\mathcal{P}_c$.

**Exercise 1** *Fill in the details of the construction above and show that these homomorphisms are surjections.*

**Exercise 2** *Show that, for $c < c'$ the kernel of $H_{K,c'} \longrightarrow H_{K,c}$ is finite, and give a precise description of this kernel. Conclude that $H_{K,c}$ is finite for any level $c$.*

**Exercise 3** *Give a precise description of the kernel of $J_K/K^* \longrightarrow H_{K,c}$ for any level $c$.*

## 1.4    Zeta functions of level $c$

Now fix a level $c$ and form
$$\zeta_{K,c}(s) \;=\; \prod_{P \,\nmid\, c_f} (1 - \frac{1}{NP^s})^{-1}$$

the product being over all nonzero prime ideals in $\mathcal{O}_K$ not dividing the ideal $c_f$. So
$$\zeta_{K,c}(s) \;=\; \sum_{0 \neq I \subset \mathcal{O}_K; (I, c_f) = 1} \frac{1}{NI^s}.$$

**Exercise 4** *Show that the Dirichlet series $\zeta K, c(s)$ has a meromorphic continuation to the half-plane $\mathcal{R}(s) > 1 - 1/n$, and in this region has only one singularity: a simple pole at $s = 1$. Give an explicit description (in terms of $K$ and $c$) of the residue at this pole.*

## 1.5    $L$-functions

We will be dealing with *level $c$ ideal class group characters*. That is, homomorphisms $\chi : H_{K,c} \to \mathbf{C}^*$, or equivalently, elements $\chi$ in the pontrjagin dual of the finite abelian group $H_{K,c}$.

Define

$$L_{K,c}(s,\chi) \quad = \quad \prod_{P \,\nmid\, c_f} (1 - \frac{\chi(P)}{NP^s})^{-1}$$

$$= \quad \sum_{0 \neq I \subset \mathcal{O}_K;(I,c_f)=1} \frac{\chi(I)}{NI^s}$$

$$= \quad \sum_{\eta \in H_{K,c}} \chi(\eta)\zeta_{K,c}(s)$$

**Exercise 5** *Using the last of the above formulas and a previous exercise, show that if $\chi$ is a nontrivial character of $H_{K,c}$, then the Dirichlet series $L_{K,c}(s,\chi)$ converges to an analytic function on the half-plane $\mathcal{R}(s) > 1 - 1/n$; specifically, it has no pole at $s = 1$. [Simple hint: If $\chi$ is a character of a finite group $H$ of order $|H|$, then $\sum_{h \in H} \chi(h) = |H|$ or $0$, depending upon whether $\chi$ is the trivial character or not.]*

## 1.6 Logs of $\zeta$-functions and $L$-functions

In this section we will be working in a small neighborhood about the point $s = 1$ in the complex plane. All the Dirichlet series introduced so far are meromorphic in such a neighborhood and we will be interested in the order of their pole or zero at $s = 1$. Equivalently, we will be interested in their asymptotic behavior for $s$ on the real interval $(1, 1+\epsilon)$ for some unspecified $\epsilon$. The equivalence relation $\simeq$ will mean "mod some regular analytic function defined locally about $s = 1$.

**Theorem 2**
$$\log \frac{1}{s-1} \quad \simeq \quad \log \zeta_K(s) \quad \simeq \quad \sum_P \frac{1}{NP^s} \quad \simeq \quad \sum_{\text{degree}(P)=1} \frac{1}{NP^s}.$$

Since
$$L_{K,c}(s,\chi) \quad = \quad \prod_{P \,\nmid\, c_f} \frac{1}{1 - \frac{\chi(P)}{NP^s}}$$

we get, similarly, that

**Theorem 3**
$$\log L_{K,c}(s,\chi) \quad \simeq \quad \sum_{P \,\nmid\, c_f} \frac{\chi(P)}{NP^s} \quad \simeq \quad \sum_{P \,\nmid\, c_f; \text{ degree}(P)=1} \frac{\chi(P)}{NP^s} \quad \simeq \quad \sum_{h \in H_{K,c}} \chi(h) \sum_{P \in h; \text{ degree}(P)=1} \frac{1}{NP^s}.$$

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By Exercise 5 we know that if $\chi$ is a nontrivial character of $H_{K,c}$ then $L_{K,c}(s, \chi)$ is analytic near $s = 1$, but we still don't know whether it vanishes at $s = 1$. Let $0 \leq m(\chi) :=$ the order of vanishing of $L_{K,c}(s, \chi)$ at $s = 1$. so

$$L_{K,c}(s, \chi) = (s - 1)^{m(\chi)} \cdot \text{regular function}$$

and

$$\log L_{K,c}(s, \chi) \simeq -m(\chi) \log \frac{1}{s - 1}.$$

# 2 Norm Class Indices

Let $L/K$ be a finite extension. Let $c = c_K$ be a level for $K$ and we'll also view it as giving a level $c_L$ for $L$ by taking $c_L := (c_{L,f}, c_{L,\infty})$ where $c_{L,f} = c_{K,f}\mathcal{O}_L$ and $c_{L,\infty} :=$ the set of real places of $L$ that lie over places in $c_{K,\infty}$.

## 2.1 Norm Class Groups

We have

$$C_K := J_K/K^* \longrightarrow H_{K,c}$$

and

$$C_L := J_L/L^* \longrightarrow H_{L,c}$$

giving rise, to:

$$C_K/N_L^K C_L = J_K/K^* N_L^K J_L \xrightarrow{\iota} H_{K,c}/N_L^K H_{L,c}.$$

**Theorem 4** *If $c$ is sufficiently large, then $\iota$ is an isomorphism*

$$C_K/N_L^K C_L \quad \overset{\iota}{\simeq} \quad H_{K,c}/N_L^K H_{L,c}.$$

Call this group the **Norm Class Group** of $L/K$, and denote its order by $X$.

 **Proof:**   I'll postpone the proof till subsection (**).

## 2.2 Logs of $L$-functions attached to Norm Class Group characters

Keep our finite extension $L/K$ and our level $c$. (Relative to this data) we will refer to a character $\chi : H_{K,c}/N_L^K H_{L,c} \to \mathbf{C}^*$ as a Norm Class Group character and note that by Theorem 3

$$\log L_{K,c}(s, \chi) \simeq \sum_{h \in H_{K,c}/N_L^K H_{L,c}} \chi(h) \sum_{P \in h; \ \text{degree}(P)=1} \frac{1}{NP^s}.$$

Now comes the fun:

$$\left(1-\sum_{\chi\neq 1}m(\chi)\right)\log\frac{1}{s-1} \; \simeq \; \zeta_K(s)+ \sum_{\chi\neq 1;\text{Norm Class Chars}} \log L_{K,c}(s,\chi) \; \simeq \; X\cdot \sum_{P\in N_L^K I_{L,c};\ \deg(P)=1} \frac{1}{NP^s}.$$

AND the RHS (a positive real number for $s$ real and $>1$) becomes only smaller if we replace it by

$$X\cdot \sum_{P\in I_{K,c};\ \deg(P)=1;\ P\ \text{splits completely in}\ L} \frac{1}{NP^s}.$$

which is

$$\frac{X}{[L:K]} \sum_{Q\in I_{L,c};\ \deg(Q)=1} \frac{1}{NQ^s} \simeq \frac{X}{[L:K]}\log\frac{1}{s-1}.$$

so

$$\left(1-\sum_{\chi\neq 1}m(\chi)\right)\log\frac{1}{s-1} \; \geq \; \frac{X}{[L:K]}\log\frac{1}{s-1} - \text{some constant}$$

on the interval $s\in(1,1+\epsilon)$.

**Corollary 5** *If $\chi$ is a nontrivial norm class character, $L_{k,c}(s,\chi)$ does not vanish at $s=1$.*

**Corollary 6**

$$|C_K/N_L^K C_L| \; \leq [L:K].$$