

NOTES FOR COHOMOLOGY LECTURE 5

5.1 The Norm Residue homomorphism. Suppose $(G, \{G_K\}, A)$ is a class formation. We have seen then that for every normal layer L/K , the cup product with the fundamental class $\alpha \in \mathcal{H}^2(L/K)$ gives an isomorphism $\alpha_{L/K}^{(r)} : \hat{\mathcal{H}}^r(G_{L/K}, \mathbb{Z}) \xrightarrow{\sim} \hat{\mathcal{H}}^{r+2}(L/K)$ for all $r \in \mathbb{Z}$. The most important new case of this isomorphism is the case $r = -2$, i.e., the isomorphisms $\alpha^{(-2)} : G_{L/K}^{\text{ab}} \xrightarrow{\sim} A_K/N_{L/K}A_L$. We will study them extensively in the section **5.3** and prove that for the K, L, M situation the following diagram is commutative:

$$(**) \quad \begin{array}{ccc} G_{M/K}^{\text{ab}} & \xrightarrow{\alpha_{M/K}^{(-2)}} & A_K/N_{M/K}A_M \\ \text{defl} \downarrow & & \downarrow \text{pass to quotient} \\ G_{L/K}^{\text{ab}} & \xrightarrow{\alpha_{L/K}^{(-2)}} & A_K/N_{L/K}A_L \end{array}$$

Taking the inverse limit vertically we find

$$(+)$$

$$G_K^{\text{ab}} = \varprojlim_L G_{L/K}^{\text{ab}} = \varprojlim_L A_K/N_{L/K}A_L,$$

both projective limits taken over the poset of finite normal extensions L of K (or over finite abelian extensions L/K ; the result is the same). For $a \in A_K$ one denotes by $(a, L/K) \in G_{L/K}^{\text{ab}}$ the element σ such that $\alpha_{L/K}^{(-2)}(\sigma) = aN_{L/K}A_L$, writing A_K multiplicatively. The map $a \mapsto (a, L/K)$ is a surjective homomorphism $A_K \rightarrow G_{L/K}^{\text{ab}}$ with kernel $N_{L/K}A_L$, called the *norm residue map* or the *reciprocity map*. The element $\sigma \in G_K^{\text{ab}}$ whose image in $G_{L/K}^{\text{ab}}$ is $(a, L/K)$ for every L normal over K is denoted by $(a, K^{\text{ab}}/K)$.

The correspondence $L \mapsto N_{L/K}A_L$ is an inclusion reversing bijection from the set of abelian layers L/K over K to a set \mathcal{N}_K of subgroups N of finite index in A_K called *norm subgroups*. A norm subgroup N of A_K corresponds to the unique level L such that $G_L/G_K^c = (N, K^{\text{ab}}/K) \subset G_K^{\text{ab}} = G_K/G_K^c$. To describe the sets \mathcal{N}_K of norm subgroups is an important problem in any class formation, since, as in (+), we have

$$(++)$$

$$G_K^{\text{ab}} = \varprojlim_L G_{L/K}^{\text{ab}} = \varprojlim_L A_K/N_{L/K}A_L = \varprojlim_{N \in \mathcal{N}_K} A_K/N.$$

The problem of describing \mathcal{N}_K is called the *Existence theorem*, because it shows the existence of abelian extensions. But before going further in this theory, let's look at some examples of class formations.

5.2 The main examples of class formations. In each of these examples the Galois group G of the formation will be the absolute Galois group G_F of a field F , and the fields are then

really fields, the finite extensions of K, L, \dots of F in a chosen separable closure. We will give the formation module A , or, equivalently, the levels A_K , in each case. However at present we can describe the fundamental classes $\alpha_{L/K}$ only in the first two examples. In each example we can say which subgroups of A_K are norm subgroups of finite abelian extensions, that is, we can state the Existence Theorem. Examples 1 and 2 are elementary but important. Examples 3 and 4 are local and global class field theory. Main goals of Math223 are to show that those formations are class formations with the stated norm subgroups, and that the global-local relations are what they are.

Example 1. $F = \mathbb{F}_p$, $G = G_{\mathbb{F}_p} \simeq \hat{\mathbb{Z}}$, $A = \mathbb{Z}$. Of course the fields are the finite extensions $K_\nu := \mathbb{F}_{p^\nu}$ of \mathbb{F}_p . The action of G on A is trivial, so $A_K = A = \mathbb{Z}$ for every K . The fundamental class $\alpha_{K_{n\nu}/K_\nu}$ is the class x of **4.1** if we identify $G_{K_{n\nu}/K_\nu}$ with the group $\{1, \tau, \dots, \tau^{n-1}\}$ there, by interpreting τ as the Frobenius $a \mapsto a^{p^\nu}$. The Norm group $N_{K_{n\nu}/K_\nu} \mathbb{Z}$ is $n\mathbb{Z}$, and the isomorphism $\alpha_{K_{n\nu}/K_\nu}^{(-2)} : G_{K_{n\nu}/K_\nu} \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z}$ takes the Frobenius to $1 \pmod{n\mathbb{Z}}$. Thus the norm subgroups of a given level are just the subgroups of finite index.

Example 2. $F = \mathbb{R}$, $G = G_{\mathbb{R}} = \text{Gal}(\mathbb{C}/\mathbb{R})$. $A = \mathbb{C}^*$. There is only one non-trivial layer, \mathbb{C}/\mathbb{R} , and $\mathcal{H}^2(\mathbb{C}/\mathbb{R}) = \mathbb{R}^*/N\mathbb{C}^* = \mathbb{R}^*/\mathbb{R}_{>0}^*$ is of order 2, and $\alpha_{\mathbb{C}/\mathbb{R}}$ is the non trivial element. Again, every subgroup of finite index is a norm subgroup.

Example 3. $F = \mathbb{Q}_p$, $A = \bar{\mathbb{Q}}_p^*$. So $A_K = K^*$ for each finite extension K of \mathbb{Q}_p . The reciprocity law map for unramified extensions is easy to describe. For L/K unramified of degree n the group $G_{L/K}$ is cyclic, generated by the K -Frobenius σ_K and we have simply $(a, L/K) = \sigma_K^{v_K(a)}$ for $a \in K^*$. The kernel of the map, the group $(K^*)^n O_K^* = \{a \in K^* | v(a) \equiv 0 \pmod{n}\}$, is the group of norms from L . In particular the units are norms from unramified extensions, and if an element is a norm from every unramified extension it is a unit. We will prove later the ‘‘Existence Theorem’’ that every subgroup of finite index is a norm subgroup. The group G_K^{ab} of the maximal abelian extension of K has an inertia subgroup I leaving fixed the maximal unramified extension of K , and a ramification group R leaving fixed the maximal tamely ramified abelian extension of K . We have $(a, K^{\text{ab}}/K) \in I \Leftrightarrow a \in O_K^*$, and $(a, K^{\text{ab}}/K) \in R \Leftrightarrow a \equiv 1 \pmod{P_K}$. In fact, the breaks in the upper numbering filtration of an abelian extension are all at integers, and we have in general, $(a, K^{\text{ab}}/K) \in (G_K^{\text{ab}})^\nu \Leftrightarrow a \equiv 1 \pmod{P_K^\nu}$. The map $a \mapsto (a, K^{\text{ab}}/K)$ is an isomorphism of K^* onto the subgroup G_K^{ab} consisting of the elements inducing an integral power of the Frobenius automorphism, the power being the valuation of a .

In part (8) of cohomology homework 2, the symbol (a, b) is $+1$ if and only if $(a, K(\sqrt{b})/K) = 1$. The non-degeneracy in part (9) of that homework shows that every subgroup of index 2 is a norm group.

Instead of $F = \mathbb{Q}_p$, we could equally well have taken $F = \mathbb{F}_p((T))$. All of the above is true for that case as well, except that we have to say that the norm subgroups are the closed subgroups of finite index, since not every subgroup of index p is closed in this case. Also the part about (a, b) applies only if $p \neq 2$.

Example 4. F a global field. $A_K = C_K = J_K/K^*$, the group of idele classes of K . The norm subgroups are the closed subgroups of finite index. In the number field case, their intersection, the kernel of the map $a \mapsto (a, K^{\text{ab}}/K)$, is not just the identity, but is the connected component of identity in C_K , usually denoted by D_K . It contains the product of the connected components of the K_v for archimedean v as a dense subgroup. Weil has suggested that to find some natural Galois-like interpretation D_K , e.g., some refinement of the map $a \mapsto (a, K^{\text{ab}}/K)$ which does not vanish on D_K , is an important problem which might even contain a clue to the Riemann hypothesis.

Of course, examples 3 and 4 are related. Let L/K be an abelian extension. For each place v of K let v' be a place of L above v . Then $G_{L_{v'}/K_v}$ is naturally a subgroup of $G_{L/K}$ independent of the choice of v' , and, writing $(a, L/K)$ for $(aK^*, L/K)$ for an idele $a = (\dots, a_v, \dots)$, we have the formula $(a, L/K) = \prod_v (a_v, L_{v'}/K_v)$ relating the global and local norm residue symbols. The right hand side of that equation is a finite product because $(a_v, L_{v'}/K_v) = 1$ if a_v is a unit and v is unramified in L . The fact that the left hand side depends only on the class of the idele a is E. Artin's reciprocity law: $\prod_v (\alpha_v, L_{v'}/K_v) = 1$ if $\alpha \in K^*$ and α_v is its image in K_v for each v . This law was discovered in the 1920's before idèles, before local class field theory, and before cohomology, in the following equivalent form. Suppose L/K abelian. then for a sufficiently large level $c = (c_f, c_\infty)$ in K (large enough so that in Mazur's Nov. 18 notation the map $C_K/N_{L/K}C_L \rightarrow I_{K,c}/P_{K,c}N_{L/K}I_{L,c}$ is an isomorphism), L/K is unramified at primes P not dividing c_f , and, denoting by σ_P the Frobenius of such a P , Artin's reciprocity law is simply

$$\prod_{P \nmid c_f} \sigma_P^{v_P(\alpha)} = 1 \quad \text{for} \quad \alpha \in P_{K,c},$$

because for the c in question $\alpha \in P_{K,c}$ implies that α is a local norm at primes P dividing c_f and at all archimedean places, so those terms of the product are 1.

It is a nice exercise to show, using the determination of the symbol for Q_p in parts (10) and (11) of cohomology homework 2, that Artin's law for quadratic extensions of Q can be expressed as $\prod_v (a, b)_v = 1$ for $a, b \in \mathbb{Q}^*$, and is trivially equivalent to Gauss's law of quadratic reciprocity (given the explicit expressions for (a, b) worked out in the homework). In the product, $v = \infty, 2, 3, 5, \dots$, and you should work out the trivial rule for $(a, b)_\infty$.

5.3 More on the norm residue symbol.

There is a nice dual description of the norm residue symbol which involves no negative dimensional cohomology. Recall that in a class formation there is given for every Galois layer L/K an "invariant" isomorphism $\text{inv}_{L/K} : \mathcal{H}^2(L/K) \xrightarrow{\sim} \frac{1}{n}\mathbb{Z}/\mathbb{Z}$ such that $\text{inv}_{L/K}(\alpha_{L/K}) = \frac{1}{n} \pmod{\mathbb{Z}}$, $n = [L/K]$. By duality for finite abelian groups, an element $\sigma \in G_{L/K}^c$ is characterized by the values $\chi(\sigma)$ for all characters $\chi : G_{L/K} \rightarrow \mathbb{C}^*$. In this way the norm residue symbol is characterized by the rule

$$(***) \quad \chi(a, L/K) = \text{inv}_{L/K}(a \cup \delta\chi)$$

for all characters χ of $G_{L/K}$. Here $\delta\chi$ is the image of χ by the usual isomorphism $\mathcal{H}^1(G_{L/K}, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \mathcal{H}^2(G_{L/K}, \mathbb{Z})$, and we interpret $a \in A_K$ as an element of $\mathcal{H}^0(L/K)$. To prove this rule is easy. Let $\sigma = (a, L/K)$ and view σ as the class of the 1-chain $[\sigma]$ in $\hat{\mathcal{H}}^{-2}(G_{L/K}, \mathbb{Z})$. Then, by definition, $\alpha_{L/K} \cup \sigma = aN_{L/K}A_L$. Cup this on the right by $\delta\chi$ and use $\sigma \cup \delta\chi = n\bar{\chi}(\sigma) \pmod{n\mathbb{Z}}$, where $\bar{\chi} : G \rightarrow \mathbb{Z}$ is a 1-cochain such that $\chi = \bar{\chi} \pmod{\mathbb{Z}}$. Then take invariants to see that $\text{inv}(\alpha_{L/K} \cdot n\bar{\chi}(\sigma)) = \text{inv}(a \cup \delta\chi)$. The left side of this equation is $\chi(\sigma)$ because $\text{inv}(\alpha_{L/K}) = \frac{1}{n} \pmod{\mathbb{Z}}$.

The commutativity of diagram (***) above follows from the formula (***) because $\chi(\text{defl } \sigma) = (\text{infl } \chi)(\sigma)$ and $\text{inv}_{M/K}(a \cup \text{infl } \delta\chi) = \text{inv}_{M/K} \text{infl}(a \cup \delta\chi) = \text{inv}_{L/K}(a \cup \delta\chi)$. Perhaps a better point of view would be to view χ as a character $G_K \rightarrow \mathbb{Q}/\mathbb{Z}$ and inv_K as a map $\mathcal{H}^2(* / K) \rightarrow \mathbb{Q}/\mathbb{Z}$. Then the formula

$$\chi(a, * / K) = \text{inv}_K(a \cup \delta\chi)$$

holds, and implies (***) for every $L \supset K$.

The relations between the reciprocity maps for different levels of a class formation are the following:

Let $K \subset L$. We have an inclusion map $i : A_K \hookrightarrow A_L$, and a homomorphism $j : G_L^{\text{ab}} \rightarrow G_K^{\text{ab}}$ induced by the inclusion $G_L \hookrightarrow G_K$. Then

Translation Theorem. $j(b, L^{\text{ab}}/L) = (N_{L/K}b, K^{\text{ab}}/K)$ for $b \in A_L$.

Transfer Theorem. $(ia, L^{\text{ab}}/L) = t(a, K^{\text{ab}}/K)$ for $a \in A_K$, where $t : G_K^{\text{ab}} = G_K/G_K^c \rightarrow G_L/G_L^c = G_L^{\text{ab}}$ is the transfer (or Verlagerung) map of cohomology homework 1.

Proof: The Translation Theorem follows from the rule $\alpha \cup \text{cores } \beta = \text{cores}(\text{res } \alpha \cup \beta)$ and the Transfer Theorem from $\text{res}(\alpha \cup \alpha') = \text{res } \alpha \cup \text{res } \alpha'$, together with the interpretations of cores and res in dimensions -2 and 0.

5.4 Establishing the axioms. To prove a formation is a class formation one usually makes use of some version of what Serre called “the ugly lemma” on page 135 of Cassels-Fröhlich.

Ugly Lemma. Let $p \geq 0$ and $q \geq -2$. Suppose a formation satisfies

- (a) For $0 < i < q$, we have $\hat{\mathcal{H}}^i(L/K) = 0$ for all normal L/K .
- (b) $|\hat{\mathcal{H}}^q(L/K)| \leq [L : K]^p$ for all cyclic L/K of prime degree.

Then the inequality (b) holds for all normal L/K .

Proof. This follows via the same kind of induction on the order of $G_{L/K}$, using the Sylow theorem, and the solvability of p -groups, that was used in 4.2 to prove the theorem on cohomological triviality. For $q \geq 1$, use the fact that (a) implies that the inflation-restriction sequence is exact in dimension q . (The cases $q = 1$ and 2 will suffice for us.) For $q = 0, -1, -2$, use the exact sequences (2), (3), (4) of the lemma following the theorem on cohomological triviality in 4.2.

Consider the following axioms for a formation.

(CF0) For L/K cyclic of prime degree, $|\mathcal{H}^2(L/K)| = [L : K]|\mathcal{H}^1(L/K)|$.

(CF1)' For cyclic L/K of prime degree, $|\mathcal{H}^1(L/K)| = 0$.

(CF2, \leq)' For cyclic L/K of prime degree, $|\mathcal{H}^2(L/K)| \leq [L : K]$.

(CF2, \leq) For every normal layer L/K , we have $|\mathcal{H}^2(L/K)| \leq [L : K]$.

(CF6)* For each field K , there is a subgroup H_K of $\mathcal{H}^2(* / K)$ and an injective homomorphism $h : H_K \rightarrow \mathbb{Q}/\mathbb{Z}$ such that:

(a) For every layer L/K we have $\text{res}_L^K H_K \subset H_L$ and $h_L(\text{res } \alpha) = [L : K]h_K(\alpha)$ for all $\alpha \in H_K$.

(b) If there exists a layer L/K of degree n then H_K contains an element of order n .

Axiom proving theorem. To prove that a formation is a class formation, it suffices to prove (CF0), either (CF1)' or (CF2, \leq)', and (CF6)*.

Proof. The case $q = 1, p = 0$ of the ugly lemma shows that (CF1)' implies (CF1). A formation satisfying (CF1) is sometimes called a *field formation*. In a field formation the inflation-restriction sequence is exact in dimension 2, and since the inflations are injective we can view them as inclusions and view $\mathcal{H}^2(L/K)$ as a subgroup of $\mathcal{H}^2(* / K) = \mathcal{H}^2(G_K, A) = \cup_{L \supset K} \mathcal{H}^2(L/K)$.

In a field formation, case $p = 1, q = 2$ of the ugly lemma shows that (CF2, \leq)' implies (CF2, \leq). These remarks and trivial juggling with inequalities show

$$(CF0) + (CF1)' \Leftrightarrow (CF0) + (CF2,\leq)' \Leftrightarrow (CF1) + (CF2,\leq).$$

To finish the proof we assume (CF1), (CF2, \leq), and (CF6)* hold and show that they imply (CF6). In fact, we show that $H_K = \mathcal{H}^2(* / K)$ so that, with $\text{inv}_K = h_K$, (CF6)* is ((CF6). Since $\mathcal{H}^2(* / K)$ is the union, over all fields L such that L/K is normal, of $\mathcal{H}^2(L/K)$, it will suffice to show for each such L that $\mathcal{H}^2(L/K) \subset H_K$. Let $n = [L : K]$. By (CF6)* (b), there is an $\alpha \in H_K$ of order n . It suffices to show $\text{res}_L^K \alpha = 0$, for then the inflation-restriction sequence

$$0 \rightarrow \mathcal{H}^2(L/K) \rightarrow \mathcal{H}^2(* / K) \rightarrow \mathcal{H}^2(* / L),$$

which is exact by (CF1), shows that $\alpha \in \mathcal{H}^2(L/K)$, and since α is of order n we conclude by (CF2, \leq) that $\mathcal{H}^2(L/K)$ is cyclic of order n generated by α , so is in H_K . To show $\text{res}_L^K \alpha = 0$ we use (CF6)* (a), which tells us that $\text{res } \alpha \in H_L$ and $h_L(\text{res } \alpha) = nh_K(\alpha) = h_K(n\alpha) = h_K(0) = 0$. Since h_L is injective, we are done.

5.5 Basic local class field theory. Let F be a local field, i.e., complete with respect to a discrete valuation with finite residue field. Let Ω be an algebraically closed field containing F and let \mathcal{K} be the set of finite separable extensions of F in Ω . Notation: For $K \in \mathcal{K}$ we let $K^{\text{unr}}, K^{\text{tame}}, K^{\text{ab}}, K^{\text{sep}}, K^{\text{alg}}$ denote the maximal unramified, tamely ramified, abelian, separable, algebraic extension of K in Ω , respectively. Write $G_K = \text{Gal}(K^{\text{sep}}/K)$ for the absolute Galois group of K . Then $G_K^{\text{ab}} = G_K/G_K^c = \text{Gal}(K^{\text{ab}}/K)$, where G_K^c is the closure of

the commutator subgroup G'_K of G_K . The aim of this section is to prove that the formation $(G_F, \{G_K\}_{K \in \mathcal{K}}, (F^{\text{sep}})^*)$ of example 3 above is a class formation.

By the ‘axiom proving theorem’ above, we have only to check axioms (CF0), (CF1)’ and (CF6)*. In the first half of cohomological homework 2, we proved (CF0) for local fields. Axiom (CF1)’ is Hilbert theorem 90 in this case, since $A_L = L^*$. For (CF6)*, let $H_{n,K} = \mathcal{H}^2(K_n/K)$, where K_n is the unramified extension of K of degree n in Ω , and put $H_K = \cup_{n=1}^{\infty} H_{n,K} = \mathcal{H}^2(K^{\text{unr}}/K)$. For each n , $G_{K_n/K}$ is cyclic of degree n , generated by the action on K_n of the Frobenius automorphism σ_K . Let $\chi_{n,K} : G_{K_n/K} \xrightarrow{\sim} \frac{1}{n}\mathbb{Z}/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$ be the character such that $\chi_{n,K}(\sigma_K) = \frac{1}{n} \pmod{\mathbb{Z}}$. By the cyclic theory, the map $a \mapsto a \cup \delta\chi$ is a surjective homomorphism $K^* \rightarrow \mathcal{H}^2(K_n/K)$ with kernel $N_{K_n/K}K_n^*$, inducing an isomorphism $K^*/NK_n^* = \hat{\mathcal{H}}^0(K_n/K) \xrightarrow{\sim} \mathcal{H}^2(K_n/K)$. Put $h_K(a \cup \delta\chi_{n,K}) = v_K(a)\chi_{n,K}(\sigma_K) = \frac{v_K(a)}{n} \pmod{\mathbb{Z}}$. This is well defined as function of a because $a \in NK_n^* \Rightarrow n|v_K(a)$. It follows from these considerations that every element of H_K is of the form $a \cup \delta\chi$ for some $a \in K^*$ and some continuous character $\chi : G_{K^{\text{unr}}/K} \rightarrow \mathbb{Q}/\mathbb{Z}$, and that the formula $h_K(a \cup \delta\chi) = v_K(a)\chi(\sigma_K)$ defines a well defined homomorphism $H_K \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$. It is injective because by construction it is injective on $H_{n,K}$ for each n . It is surjective because the image contains $\frac{1}{n} \pmod{\mathbb{Z}}$ for every n .

Finally we must show for all $L \supset K$ that

$$h_L(\text{res}_L^K(a \cup \delta\chi)) = [L : K]h_K(a \cup \delta\chi), \quad \text{i.e.} \quad v_L(a) \text{res } \chi(\sigma_L) = [L : K]v_K(a)\chi(\sigma_K),$$

for $a \in K^*$ and χ a unramified character of G_K and $\text{res } \chi$ its restriction to G_L . The last formula holds because $v_L(a) = e v_K(a)$, where e is the ramification index of L/K , and $(\text{res } \chi)(\sigma_L) = f\chi(\sigma_K)$, where f is the residue degree of L/K , because the action of σ_L on K^{unr} is that of σ_K^f . And $ef = [L : K]$.

Thus $H_K = \mathcal{H}^2(* / K)$ and $\text{inv}_K = h_K$ is an isomorphism $\mathcal{H}^2(* / K) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$.

5.6 The local existence theorem. In this section we want to prove the number field case of

Existence Theorem of local class field theory. For K local, the norm subgroups of K^* are the open subgroups of finite index.

In the number field case (K a finite extension of \mathbb{Q}_p) we don’t need to say open. Every subgroup of finite index is open, for if m is an integer > 0 . then the binomial series for $(1 + mx)^{\frac{1}{m}}$ converges for $|x| < |p|^{\frac{1}{p-1}}$. (Use $\text{ord}_p n! = \sum_{i=1}^{\infty} [n/p^i] < \sum_{i=1}^{\infty} n/p^i = \frac{n}{p-1}$. Exercise: Using the Herbrand quotient or otherwise, prove that the index $(K^* : (K^*)^m)$ is equal to $\frac{n|\mu_m(K)|}{|n|_K}$, where $| \cdot |_K$ is the normed absolute value. Example: $(\mathbb{Q}_2 : (\mathbb{Q}_2^*)^2) = \frac{2 \cdot 2}{1/2} = 8$.)

In the function field case (K a finite extension of $\mathbb{F}_p((T))$), a subgroup of index m is open if $(p, m) = 1$, but not all subgroups of index p are open because $K^*/(K^*)^p$ is an infinite compact abelian group killed by p .

We will use the following properties of the set \mathcal{N}_K of norm subgroups in a class formation. The proofs are an easy exercise if we recall that for an arbitrary normal layer L/K , there is a surjective homomorphism $a \mapsto (a, L/K)$ from A_K to $G_{L/K}^{\text{ab}} = G_{L/K}/G_{L/K}^c$ and these homomorphisms are compatible in the sense that the diagram (***) of **5.1** is commutative, that is, for $K \subset L \subset M$, the image of $(a, M/K)$ in $G_{L/K}^{\text{ab}}$ is $(a, L/K)$.

(i) If $K \subset L$ then $N_{L/K}A_L = N_{L'/K}A_{L'}$, where L' is the maximal field $K \subset L' \subset L$ such that L'/K is abelian. In particular, every norm subgroup is the norm subgroup of an abelian extension, and for an arbitrary layer L/K we have $(A_K : N_{L/K}A_L) \leq [L : K]$, with equality if and only if L/K is abelian. (This is clear if L/K is Galois because in that case $G_{L/K}^{\text{ab}} = G_{L'/K}^{\text{ab}} = G_{L'/K}$, hence $(A_K : N_{L/K}A_L) = (A_K : N_{L'/K}A_{L'})$. Since $N_{L'/K}A_{L'} \subset N_{L/K}A_L$ by transitivity of norm, the two subgroups are equal as claimed. The non-normal case is an easy exercise in Galois theory. One shows that the maximal abelian extension L' of K in L is that of K in the Galois closure of L .)

(ii) A subgroup of A_K containing a norm subgroup is a norm subgroup (If $N \subset A_K$ and $N \supset N_{L/K}A_L$ we can assume L/K is abelian by (i). Let H be the subgroup of $G_{L/K} \simeq A_K/N_{L/K}A_L$ corresponding to $N/N_{L/K}A_L$, and let M be the intermediate field such that $G_{M/K} = G_{L/K}/H$. Then N is the kernel of the map $a \mapsto (a, M/K)$ from A_K to $G_{M/K}$, so $N = N_{M/K}A_M$ is a norm subgroup.)

(iii) If H is a norm subgroup of A_L then $N_{L/K}H$ is a norm subgroup of A_K (because the norms are transitive).

Proof of the theorem for K a finite extension of \mathbb{Q}_p : Let $N \subset K^*$ be a subgroup of finite index, say $(K^* : N) = m$. Let $L = K(\zeta)$, where ζ is a primitive m -th root of 1 in Ω . We know that $L^*/(L^*)^m$ is finite. Let $M = L((L^*)^{\frac{1}{m}})$, the field obtained by adjoining to L all the elements of Ω whose m -th power is in L . By Kummer theory, M/L is abelian and $G_{M/L}$ is the Cartier dual of $L^*/(L^*)^m$, and in particular, is killed by m . By class field theory, $G_{M/L} \xrightarrow{\sim} L^*/N_{M/L}M^*$, and since this group is killed by m , we have $N_{M/L}M^* \supset (L^*)^m$. But these two subgroups of L^* both have the same index, because the order of $G_{L/M}$ is $(L^* : (L^*)^m)$ by duality. Hence $(L^*)^m = N_{M/L}M^*$ is a norm subgroup of L^* . By (iii) above, $N_{L/K}(L^*)^m$ is a norm subgroup of K^* . And since $N_{L/K}(L^*)^m = (N_{L/K}L^*)^m \subset (K^*)^m \subset N$, it follows by (i) above that N is a norm subgroup of K^* as was to be shown.