Brauer and Artin Reciprocity

December 21, 2008

Contents

1 H-functors

Let K_o be a field, and $\bar{K}_o/K_o$ an algebraic closure. We will be considering intermediate Galois field extensions L/K, with $K_o \subset K \subset L \subset \bar{K}_o$ where L/K_o is of finite degree. These form a category in the evident way with morphisms $(L/K) \to (L'/K')$ just K_o-homomorphisms $L \to L'$ sending K to K'. We have our general notion of *class formation* and I would like to single out some specific axioms that our class formations satisfy, these being particularly relevant in the constructions below. There are, of course, other axioms that our examples enjoy but the following will be most relevant. To have terminology for this, define provisionally, an H**-functor** on this category of extensions L/K with values in some abelian category $\mathcal{A}$ to be an $\mathcal{A}$-valued functor $(L/K) \mapsto H(L/K)$ such that given a triple $K \subset L \subset M$ with L/K and M/K in our category—so that we have morphisms

$$(L/K) \to (M/K) \to (M/L)$$

in our category of field extensions—we have the following **left-exactness property:** the induced sequence of $\mathcal{A}$-morphisms

$$0 \to H(L/K) \to H(M/K) \to H(M/L)$$

is exact. We can call the first induced morphism *inflation* and the second one *restriction* as usual, and this sequence becomes the *inflation/restriction* exact sequence

$$0 \to H(L/K) \xrightarrow{\text{Inf}} H(M/K) \xrightarrow{\text{Res}} H(M/L)$$

.

Note that since for any triple in our category $M/L/K$ the induced homorphism $H(L/K) \hookrightarrow H(M/K)$ is injective we may simply identify $H(L/K)$ with the image subgroup in $H(M/K)$. We may then define

$$H(*/K) := \lim_{K \subset L \subset \bar{K}_o} H(L/K) = \cup_{K \subset L \subset \bar{K}_o} H(L/K)$$

and note that it is functorial in the sense that given any base change K'/K we have the induced homomorphism

$$Res_K^{K'} : H(\ /K) \to H(\ /K').$$

1.1 Trace Mappings

Let us say that our H-functor has a **Trace mapping** if we are given a $\mathcal{A}$-morphism,

$$\mathrm{Cor} = \mathrm{Cor}_{L/K} = \mathrm{Cor}_{M/L/K} : H(M/L) \to H(M/K)$$

that is

- functorial in the initial triple (in the evident sense),
- that commutes with inflation,

$$\mathrm{Inf}_M^{M'} \cdot \mathrm{Cor}_{M/L/K} \quad = \quad \mathrm{Cor}_{M'/L/K} \cdot \mathrm{Inf}_M^{M'}.$$

- that has the property that the composition

$$H(M/K) \xrightarrow{Res} H(M/L) \xrightarrow{Cor} H(M/K)$$

  is multiplication by $[L : K]$.

1.2 Examples

- Thanks to Hilbert's Theorem 90 and our general theory of cohomology of groups we have that for any K_o,

$$(L/K) \mapsto H(L/K) := H^2(\mathrm{Gal}(L/K), L^*)$$

  is indeed an H-functor with Trace Mapping (with the standard corestriction = "Cor"). We will be dealing with this in the local case (i.e., where $K_o = \mathrm{Q}_p$) or the global case (i.e., where $K_o = \mathrm{Q}$).

- By the end of these notes we will have done all the preparatory work to prove that in the global case

$$(L/K) \mapsto H(L/K) := H^2(\mathrm{Gal}(L/K), C_L)$$

  is an H-functor with Trace Mapping.

- Throughout these notes we will be dealing with what we will be calling *the Brauer H-functor*:

$$(L/K) \mapsto H(L/K) := H^2(\mathrm{Gal}(L/K), J_L) = \bigoplus_{v \text{ place of } K} H^2(\mathrm{Gal}(L_w/K_v), L^*)$$

  (with notation conventions as in section 2 below). which is also an H-functor. It has a natural corestriction operation given as the direct sum of the local corestrictions, but this does *not* satisfy the second requirement of a Trace mapping given in the definition the sense described above, but (as we will be discussing) has the property that the composition

$$H(M/K)\xrightarrow{Res}H(M/L)\xrightarrow{Cor}H(M/K)$$

is multiplication—*not by the degree* $[L:K]$, *but*—by the *local degree function* (see Definition 6 below) of L/K operating on

$$H(M/K) = \bigoplus_{v \text{ place of } K} H^2(\mathrm{Gal}(M_w/K_v), L^*).$$

1.3 Invariant Functionals

Definition 1 *Let* $L/K \mapsto H(L/K)$ *be an* H*-functor to the category of abelian groups. An* **invariant functional** *on this* H*-functor is a collection of homomorphisms*

$$(L/K) \mapsto inv_K : H(L/K) \hookrightarrow \mathrm{Q/Z}$$

that

1. *commutes with inflation and more generally, if* $(L,K) \to (L',K')$ *is a morphism in our category, and if* $\iota : H(L/K) \to H(L'/K')$ *is the induced homomorphism, then*

$$inv_{K'} \cdot \iota \;=\; [K':K] \cdot inv_K.$$

2. *for any* K'/K *we have:*

$$inv_K \cdot \mathrm{Cor}_{K'/K} \;=\; inv_{K'}.$$

1.4 Local Example

The local theory gives us (an *invariant functional*)

$$inv_K : H^2(\mathrm{Gal}(L/K), L^*) \hookrightarrow \mathrm{Q/Z},$$

that we have already studied in depth. In the special case where we have a character $\chi : \mathrm{Gal}(L/K) \hookrightarrow \mathrm{Q/Z}$ that is an imbedding (implying, of course, that L/K is cyclic) consider

- the cohomology class $\delta\chi \in H^2(\mathrm{Gal}(L/K), \mathrm{Z})$ where δ is the connecting homomorphism attached to the exact sequence of $\mathrm{Gal}(L/K)$-modules

$$0 \to \mathrm{Z} \to \mathrm{Q} \to \mathrm{Q/Z} \to 0$$

(these being given trivial $\mathrm{Gal}(L/K)$ -action) and

- for $a \in K^* = H^0(\mathrm{Gal}(L/K), L^*)$ let $\hat{a} \in \hat{H}^0(\mathrm{Gal}(L/K), L^*) = K^*/N_{L/K}L^*$ denote its image, and form the cup-product

$$\hat{a} \cup \delta\chi \ \in \ H^2(\mathrm{Gal}(L/K), L^*)$$

  noting that:

  - *Every* element in $H^2(\mathrm{Gal}(L/K), L^*)$ is expressible as $\hat{a} \cup \delta\chi$ for some $a \in K^*$, and
  - we have the formula:

    Theorem 1

$$inv_K(\hat{a} \cup \delta\chi) \ = \ \chi(a, L/K).$$

2 Global Example: the Brauer H-functor

Here we put $K_o = \mathrm{Q}$, and from now, till the end of these notes, set: $H(L/K) = H^2(\mathrm{Gal}(L/K), J_L)$. For every place v of K, choose a place w of L lying above v. The functor

$$(L/K) \ \mapsto \ H(L/K) := \ H^2(\mathrm{Gal}(L/K), J_L) \ = \ \bigoplus_{v \text{ place of } K} H^2(\mathrm{Gal}(L_w/K_v), L_w^*)$$

is an H-functor. This H-functor does not have a trace mapping as axiomatized above, but does have the direct sum of its local corestrictions.

Neither does the H-functor $(L/K) \mapsto H(L/K) = H^2(\mathrm{Gal}(L/K), J_L)$ have an *invariant functional* as axiomatized by the *two* items in Definition 1 above but, again, it does have a version of it as follows: define our global functional, $I_{L/K}$, to be the sum of the local invariants. Specifically, if

$$c \in H(L/K) \ = \ H^2(\mathrm{Gal}(L/K), J_L) \ = \ \bigoplus_{v \text{ place of } K} H^2(\mathrm{Gal}(L_w/K_v), L_w^*)$$

and if we denote by $c_v \in H^2(\mathrm{Gal}(L_w/K_v), L_w^*)$ the v-th coordinate of c in the sense that

$$c = \bigoplus_{v \text{ place of } K} c_v \ \in \ \bigoplus_{v \text{ place of } K} H^2(\mathrm{Gal}(L_w/K_v), L_w^*),$$

then

$$I_{L/K}(c) := \sum_{v \text{ place of } K} inv_v c_v \ \in \ \mathrm{Q/Z}.$$

This functional has, at least, these properties:

Lemma 1 *1. $I_{L/K}$ commutes with inflation.*

2. for any K'/K we have the equality $inv_K \cdot \mathrm{Cor}_{K'/K} \ = \ inv_{K'}$.

An element $c \in H(L/K)$ (such an element we may call a ' *"Brauer class"*) is completely determined by the local invariants, e.g., by the vector

$$\bigoplus_{v \text{ place of } K} inv_v(c_v) \ \in \ \bigoplus_{v \text{ place of } K} \mathrm{Q/Z}.$$

Since we know that $H^1(\mathrm{Gal}(L/K), C_L) = 0$, the exact sequence

$$0 \to L^* \to J_L \to C_L \to 0$$

of $\mathrm{Gal}(L/K)$-modules induces an exact sequence

$$0 \to H^2(\mathrm{Gal}(L/K), L^*) \to H(L/K) \to H^2(\mathrm{Gal}(L/K), C_L)$$

allowing us to identify $H^2(\mathrm{Gal}(L/K), L^*)$ as a subgroup of $H(L/K)$; call this the subgroup of **principal Brauer classes**. The H-functor $(L/K) \mapsto H^2(\mathrm{Gal}(L/K), L^*)$, then, is a sub-$H$-functor of $(L/K) \mapsto H(L/K)$.

3 Reciprocity

In this section, again, let L/K be global, and we fix attention on the H-functor $H(L/K) = H^2(\mathrm{Gal}(L/K), J_L)$.

Definition 2 *Say that L/K satisfies* **Brauer Reciprocity** *if $I_{L/K}$ vanishes on every principal Brauer class. That is, for every $c \in H^2(\mathrm{Gal}(L/K), L^*) \subset H(L/K)$ we have:*

$$I_{L/K}(c) \;=\; 0.$$

Equivalently, we might write the above equation as:

$$\sum_{v \text{ place of } K} inv_v c_v \;=\; 0 \;\in\; \mathrm{Q/Z}.$$

We will compare this with Artin reciprocity: for L_w/K_v a local abelian Galois extension and $a_v \in K_v^*$ recall the notation

$$(a_v, L_w/K_v) \in \mathrm{Gal}(L_w/K_v)$$

for the Artin symbol. If L_w/K_v comes—with evident notation—from an abelian global extension L/K we may view $\mathrm{Gal}(L_w/K_v)$ as canonically imbedding in $\mathrm{Gal}(L/K)$ and take this Artin symbol to be an element in the global Galois group $\mathrm{Gal}(L/K)$.

Definition 3 *Let L/K be an abelian Galois extension. Say that L/K satisfies* **Artin Reciprocity** *if for every $a \in K^*$ we have*

$$\prod_{v \text{ place of } K} (a_v, L_w/K_v) \;=\; 1 \;\in\; \mathrm{Gal}(L/K).$$

Lemma 2 *If L/K is cyclic, Artin and Brauer reciprocity for L/K are equivalent.*

Proof: If $\chi : \mathrm{Gal}(L/K) \hookrightarrow \mathrm{Q/Z}$ is an imbedding, then any $c \in H^2(\mathrm{Gal}(L/K), L^*)$ is of the form $c = \hat{a} \cup \delta\chi$ for some $a \in K^*$. For v a place of K, let $a_v \in K_v^*$ denote the image of a, and $\chi_v : \mathrm{Gal}(L_w/K_v) \hookrightarrow \mathrm{Q/Z}$ the restriction of χ. We have the formula:

$$I_{L/K}(c) = \sum_{v \text{ place of } K} inv_v c_v = \sum_{v \text{ place of } K} inv_v(\hat{a}_v \cup \delta\chi_v.$$

By Theorem 1 we then get

$$I_{L/K}(c) = \sum_{v \text{ place of } K} \chi_v(a_v, L_w/K_v) = \chi(\prod_{v \text{ place of } K} (a_v, L_w/K_v)) \in \mathrm{Q/Z}.$$

Since χ is injective, we have that

$$\prod_{v \text{ place of } K} (a_v, L_w/K_v) = 1 \in \mathrm{Gal}(L/K)$$

if and only if $I_{L/K}(c) = 0 \in \mathrm{Q/Z}$.

Lemma 3 *Brauer reciprocity for all Galois field extensions M/Q implies Brauer reciprocity for all (global) Galois field extensions L/K.*

Proof: Consider L/K a Galois field extension in our category (i.e., L is of finite degree over Q) and fix attention on a principal Brauer class $c \in H^2(\mathrm{Gal}(L/K), L^*) \subset H(L/K)$. Let M be, say, the Galois closure of L over Q. We have

$$I_{L/K}(c) = I_{M/K}(Inf_L^M(c)) = I_{M/\mathrm{Q}}(Cor_{M/K/\mathrm{Q}} \cdot Inf_L^M(c)) = 0,$$

where the first equality follows from condition (1) of Lemma **??**, and the second equality from condition (2) of Lemma **??**.

Definition 4 *Let L/K and L'/K be objects of our category (same base field K) and let $c \in H(L/K)$, and $c' \in H(L'/K)$ be Brauer classes. Say that c is* **equivalent to** *c' (in notation $c \simeq c'$) if c and c' have a common image in $H(LL'/K)$, or equivalently, in the direct limit $H(\ /K)$; or—in formulas–if:*

$$Inf_L^{LL'} c = Inf_{L'}^{LL'} c'.$$

Clearly, if $c \simeq c'$, then $I_{L/K}(c) = I_{L'/K}(c')$.

Definition 5 *Let $c \in H(L/K)$. By the* **"denominator"** *of c we mean the function*

$$v \mapsto d_v := \mathrm{order}(c_v) = \mathrm{order}(inv_v(c_v))$$

for v a place of K.

Definition 6 *By the* **"local degree function"** *of L/K we mean the function $v \mapsto [L_w : K_v]$ for v a place of K.*

It is clear what is meant by a “denominator” **dividing** a “local degree function.”

Lemma 4 *1. Let $c \in H(L/K)$ and let L'/K be an element of our category such that the local degree function of L'/K is a multiple of the “denominator” of c.*

Then there is a unique $c' \in H(L'/K)$ such that $c \simeq c'$.

2. If, under the hypotheses of the previous item, c were a principal Brauer class, then there is a unique principal Brauer class $c' \in H(L'/K)$ such that $c \simeq c'$.

Proof.

1. We make use of the inflation/restriction sequence

$$0 \to H(L'/K) \to H(LL'/K) \to H(LL'/L').$$

Consider $\tilde{c} := Inf_{L/K}^{LL'/K}(c) \in H(LL'/K)$. Form $Res_{LL'/K}^{LL/L'}(\tilde{c})$ noting that under our assumptions all its local invariants vanish, so $\tilde{c}$ is the image of a unique element of $H(L'/K)$.

2. Starting with a principal Brauer class $c \in H^2(\mathrm{Gal}(L/K), L^*) \subset H(L/K)$ and following the above procedure using the inflation/restriction sequence (of subgroups of the above)

$$0 \to H^2(\mathrm{Gal}(L'/K), L^{'*}) \to H^2(\mathrm{Gal}(LL'/K), (LL^{'})^*) \to H^2(\mathrm{Gal}(LL'/L'), (LL^{'})^*),$$

we obtain a unique (principal) Brauer class $c' \in H^2(\mathrm{Gal}(L'/K), L^{'*})$ equivalent to c.

4 Cyclotomic extensions over Q

4.1 Cyclotomic recall

Let F be any field of characteristic 0 and $\bar{F}$ an algebraic closure of F. We have: —

- $\mu(\bar{F})$:= the group of roots of unity in $\bar{F}$; equivalently: the torsion subgroup of $\bar{F}^*$, and
- for ℓ a prime number, $\mu(\bar{F}; \ell)$:= the group of ℓ-power roots of unity in $\bar{F}$; equivalently: the ℓ-primary component of $\mu(\bar{F})$.

So,

$$\mathrm{Q}/\mathrm{Z} \simeq \mu(\bar{F}) \quad = \quad \prod_{\ell} \mu(\bar{F}; \ell) \quad \simeq \quad \prod_{\ell} \mathrm{Q}_\ell/\mathrm{Z}_\ell.$$

Let

$$F(\mathrm{cycl}) : = \; F(\mu(\bar{F})) \; \subset \; \bar{F},$$

and

$$F(\text{cycl};\ell) := F(\mu(\bar{F}:\ell)) \subset F(\text{cycl}) \subset \bar{F}.$$

So,

$$\text{Gal}(F(\text{cycl})/F) \hookrightarrow \text{Aut}(\mu(\bar{F})) = \hat{\text{Z}}^*,$$

and

$$\text{Gal}(F(\text{cycl};\ell)/F) \hookrightarrow \text{Aut}(\mu(\bar{F};\ell)) = \text{Z}_\ell^*,$$

and these injections are compatible with the canonical identifications of $\hat{\text{Z}}^*$ with $\prod_\ell \text{Z}_\ell^*$, and $\text{Gal}(F(\text{cycl})/F)$ with $\prod_\ell \text{Gal}(F(\text{cycl};\ell)/F)$.

Thanks to Gauss, these injections are isomorphisms if $F = \text{Q}$ (also if $F = \text{Q}_p$).

Setting $F = \text{Q}$ let us pin down our conventions, and describe the canonical (inverse) to the isomorphism above,

$$\tau : \hat{\text{Z}}^* \xrightarrow{\simeq} \text{Gal}(\text{Q}(\text{cycl})/\text{Q}),$$

by the formula

$$\tau(u) = \{\zeta \mapsto \zeta^{\sigma(u)} := \zeta^{u^{-1}}\} \in \text{Gal}(\text{Q}(\text{cycl})/\text{Q}),$$

where I hope the meaning of this is clear: u is an arbitrary element of $\hat{\text{Z}}^*$ and ζ is an arbitrary root of unity (i.e., element in $\mu(\bar{\text{Q}})$). And we have the "local counterparts"

$$\tau_\ell : \text{Z}_\ell^* \xrightarrow{\simeq} \text{Gal}(\text{Q}(\text{cycl};\ell)/\text{Q}) = \text{Gal}(\text{Q}_\ell(\text{cycl};\ell)/\text{Q}_\ell),$$

by the formula

$$\tau_\ell(u_\ell) = \{\zeta \mapsto \zeta^{\tau(u)} := \zeta^{u^{-1}}\} \in \text{Gal}(\text{Q}(\text{cycl})/\text{Q}),$$

where

$$\tau_\ell(u_\ell) = \{\zeta\{\ell\} \mapsto \zeta\{\ell\}^{\tau_\ell(u_\ell)} := \zeta\{\ell\}^{u_\ell^{-1}}\} \in \text{Gal}(\text{Q}(\text{cycl}:\ell)/\text{Q}) = \text{Gal}(\text{Q}_\ell(\text{cycl};\ell)/\text{Q}_\ell),$$

where u_ℓ is an ℓ-adic unit and $\zeta\{\ell\}$ is an arbitrary *ℓ-power* root of unity (i.e., element in $\mu(\bar{\text{Q}};\ell)$).

Any root of unity ζ can be factored uniquely as a product of ℓ-power roots of unity for all primes ℓ, and so we have a (unique) formula of the following sort:

$$\zeta = \prod_\ell \zeta\{\ell\}$$

where $\zeta\{\ell\}$ is an ℓ-power root of unity. With this convention, we also have the formula:

$$\zeta^{\tau(u)} = \prod_\ell \zeta\{\ell\}^{\tau_\ell(u_\ell)} = \prod_\ell \zeta\{\ell\}^{u_\ell^{-1}}.$$

4.2 The Local degree function of Cyclotomic Extensions, and Brauer reciprocity.

Exercise 1 *For $p \leq \infty$, let $p \mapsto D_p$ be a function to positive integers, where $D_\infty \leq 2$. and such that for all but finitely many p, $D_p = 1$. There exists a cyclic cyclotomic extension $K \subset \mathrm{Q}(\mu_r)$ such that for all $p \leq \infty$, $[K_v : \mathrm{Q}_p]$ is a multiple of D_p.*

Corollary 2 *To prove Brauer reciprocity for all L/K (see definition 2 above) it suffices to prove it for cyclic cyclotomic extensions K/Q.*

Proof. By Lemma 3 it suffices to show Brauer reciprocity for K/Q, and by Lemma 4 and Exercise 1 above, we may reduce our consideration to cyclic cyclotomic extensions.

5 The idele group and the idele class group of Q

5.1 The idele group of Q

Let J_Q be the group of Q-ideles with its usual topology and view Q^* as a topological group with (also as usual) its discrete topology. We'll write out our ideles as:

$$\alpha = (\alpha_\infty; \alpha_2, \alpha_3, \alpha_5, \alpha_7, \alpha_{11}, \ldots, \alpha_\ell, \ldots).$$

Define the continuous surjective homomorphism

$$\mathit{Field} : J_\mathrm{Q} \quad \longrightarrow \quad \mathrm{Q}^*$$

by the formula

$$\alpha \mapsto \mathit{Field}(\alpha) := \quad \mathrm{sign}(\alpha_\infty) \cdot \prod_\ell \ell^{v_\ell(\alpha_\ell)} \quad \in \quad \mathrm{Q}^*.$$

The kernel of *Field* consists of ideles α such that α_∞ is positive anbd α_ℓ is a unit for all primes ℓ. Set

$$D_\mathrm{Q} := \mathbf{R}^{>0} \hookrightarrow J_\mathrm{Q}$$

where the inclusion is given by

$$\alpha_\infty \mapsto (\alpha_\infty; 1, 1, \ldots, 1, \ldots)$$

and note that D_Q is the connected component containing the identity in J_Q. The kernel of *Field* is

$$D_\mathrm{Q} \times \hat{\mathrm{Z}}^* = D_\mathrm{Q} \times \prod_\ell \mathrm{Z}_\ell^* \quad \subset J_\mathrm{Q}.$$

(The injection of Z_ℓ^* in J_Q is what you will have guessed; namely: $\alpha_\ell \mapsto (1; 1, 1, \ldots, 1, \alpha_\ell, 1, \ldots)$).

5.2 The global (surjective) homomorphism $\sigma : J_{\mathrm{Q}} \quad \rightarrow \quad \hat{\mathrm{Z}}^* \quad = \quad \mathrm{Gal}(\mathrm{Q(cycl)/Q})$

We have the exact sequence:

$$0 \quad \rightarrow \quad D_{\mathrm{Q}} \times \hat{\mathrm{Z}}^* \rightarrow J_{\mathrm{Q}} \quad \stackrel{\textit{Field}}{\longrightarrow} \quad \mathrm{Q}^* \quad \rightarrow \quad 0,$$

and from this we induce an exact sequence:

$$0 \quad \rightarrow \quad D_{\mathrm{Q}} \quad \rightarrow \quad C_{\mathrm{Q}} \quad \rightarrow \quad \hat{\mathrm{Z}}^* \quad \rightarrow \quad 0.$$

Putting it in an equivalent way, we have—reading from left to right—-a surjective homomorphism, and isomorphisms,

$$J_{\mathrm{Q}} \quad \rightarrow \quad C_{\mathrm{Q}}/D_{\mathrm{Q}} \quad \stackrel{\simeq}{\longrightarrow} \quad \hat{\mathrm{Z}}^* \quad = \quad \mathrm{Gal}(\mathrm{Q(cycl)/Q})$$

given by:

$$\alpha \quad \mapsto \quad \{\text{class of } \alpha \ \in C_{\mathrm{Q}}/D_{\mathrm{Q}}\} \quad \mapsto \quad \sigma(\alpha) \quad \in \quad \mathrm{Gal}(\mathrm{Q(cycl)/Q}).$$

5.3 The local homomorphism $\sigma_p : \mathrm{Q}_p^* \quad \rightarrow \quad \hat{\mathrm{Z}}^* \quad = \quad \mathrm{Gal}(\mathrm{Q(cycl)/Q})$

Fix a prime number p (or take $p = \infty$) and consider

$$\mathrm{Q}_p^* \quad \hookrightarrow J_{\mathrm{Q}} \quad \stackrel{\sigma}{\longrightarrow} \quad \mathrm{Gal}(\mathrm{Q(cycl)/Q}).$$

Call the composition of the continuous homomorphisms displayed above

$$\sigma_p : \mathrm{Q}_p^* \rightarrow Gal(\mathrm{Q(cycl)/Q}),$$

noting that it too is continuous. Its formula is given as follows: for an $\alpha_p \in \mathrm{Q}_p^*$, set $v_p(\alpha_p) = m$, and put

$$u_p := \alpha_p/p^m \in \mathrm{Z}_p^*.$$

The image of α_p in

$$\hat{\mathrm{Z}}^* = \prod_{\ell \neq p} \mathrm{Z}_\ell^* \times \mathrm{Z}_p^*$$

under the projection

$$J_{\mathrm{Q}} \rightarrow \prod_{\ell} \mathrm{Z}_\ell^*$$

is the same as the image of

$$\alpha_p/p^m = (p^{-m}; p^{-m}, p^{-m}, \ldots, p^{-m}, u_p, p^{-m}, \ldots).$$

First suppose that $p \neq \infty$; this image in

$$\hat{\mathrm{Z}}^* = \prod_{\ell \neq p} \mathrm{Z}_\ell^* \times \mathrm{Z}_p^* \quad = \quad \prod_{\ell \neq p} \mathrm{Gal}(\mathrm{Q}(\mathrm{cycl}; \ell)/\mathrm{Q}) \times \mathrm{Gal}(\mathrm{Q}(\mathrm{cycl}; p)/\mathrm{Q}) \quad = \quad \mathrm{Gal}(\mathrm{Q(cycl)/Q})$$

has coordinates:

$$\sigma_p(\alpha_p) \quad = \quad (Frob_p^m, Frob_p^m, \ldots, Frob_p^m, \ldots; \tau_p(u_p^{-1}))$$

in

$$\prod_{\ell \neq p} \mathrm{Gal}(\mathrm{Q}(\mathrm{cycl}; \ell)/\mathrm{Q}) \times \mathrm{Gal}(\mathrm{Q}(\mathrm{cycl}; p)/\mathrm{Q}) \quad = \quad \mathrm{Gal}(\mathrm{Q}(\mathrm{cycl})/\mathrm{Q}).$$

Or, to view $\sigma_p(\alpha_p)$ as an automorphism of $\mu(\mathrm{Q})$ write any $\zeta \in \mu(\mathrm{Q})$ as a product

$$\zeta = \zeta\{\text{prime to } p\} \cdot \zeta\{p\}$$

(with the evident meaning here) and we have:

$$\zeta^{\sigma_p(\alpha_p)} \quad = \quad \zeta\{\text{prime to } p\}^{Frob_p^m} \cdot \zeta\{p\}^{u_p^{-1}}.$$

In the case where $p = \infty$ we have that

$$\sigma_\infty : \mathbf{R}^* \to \{\pm 1\} \quad \subset \quad \mathrm{Gal}(\mathrm{Q}(\mathrm{cycl})/\mathrm{Q})$$

is given by the *sign*; i.e., $\sigma_\infty(\alpha_\infty) = \alpha_\infty/|\alpha_\infty|$.

Now is a good time to note that (for $p \leq \infty$) when we consider the cyclotomic extension $\mathrm{Q}_p(\mathrm{cycl}) \subset \bar{\mathrm{Q}}_p$ of Q_p we have—as with all cyclotomic extensions— the canonical imbedding

$$\mathrm{Gal}(\mathrm{Q}_p(\mathrm{cycl})/\mathrm{Q}_p) \quad \hookrightarrow \quad \hat{\mathrm{Z}}^* \quad = \quad \mathrm{Gal}(\mathrm{Q}(\mathrm{cycl})/\mathrm{Q}),$$

and that our homomorphism $\sigma_p : \mathrm{Q}_p^* \to \hat{\mathrm{Z}}^*$ is injective and has image lying densely in the subgroup $\mathrm{Gal}(\mathrm{Q}_p(\mathrm{cycl})/\mathrm{Q}_p)$ in $\hat{\mathrm{Z}}^*$.

We use the same letter to denote the homomorphism

$$\sigma_p : \mathrm{Q}_p^* \to \mathrm{Gal}(\mathrm{Q}_p(\mathrm{cycl})/\mathrm{Q}_p) \quad \hookrightarrow \quad \hat{\mathrm{Z}}^* \quad = \quad \mathrm{Gal}(\mathrm{Q}(\mathrm{cycl})/\mathrm{Q}).$$

5.4 The relation between local and global "sigmas"

Exercise 2

$$\sigma(\alpha) \quad = \quad \lim_{C \to \infty} \prod_{\ell \leq C} \sigma_\ell(\alpha_\ell) \quad = \quad \prod_{\ell} \sigma_\ell(\alpha_\ell) \quad \in \quad \mathrm{Gal}(\mathrm{Q}(\mathrm{cycl})/\mathrm{Q}).$$

6 Norm class groups for cyclotomic extensions

6.1 The kernel of σ

Now let us *fix* a global cyclotomic field; that is, a subfield field $K \subset \mathrm{Q}(\mu_r)$ for some positive integer r. Consider the norm mapping on idele groups:

$$J_K \stackrel{N_{K/\mathrm{Q}}}{\longrightarrow} J_{\mathrm{Q}}.$$

Proposition 1 *The composition*

$$J_K \stackrel{N_{K/\mathrm{Q}}}{\longrightarrow} J_{\mathrm{Q}} \stackrel{\sigma}{\longrightarrow} \mathrm{Gal}(\mathrm{Q}(\mathrm{cycl})/\mathrm{Q}) \rightarrow \mathrm{Gal}(K/\mathrm{Q})$$

vanishes.

Proof. Start with an arbitrary $\beta \in J_K$, and we must show that $\sigma(N_{K/\mathrm{Q}}(\beta)) \in \mathrm{Gal}(\mathrm{Q}(\mathrm{cycl})/\mathrm{Q})$ fixes the subfield $K \subset \mathrm{Q}(\mathrm{cycl})$.

Write

$$N_{K/\mathrm{Q}}(\beta) = \alpha = (\ldots, \alpha_\ell, \ldots) \in J_{\mathrm{Q}}$$

where

$$\alpha_\ell = \prod_{v \mid \ell} N_{K_v/\mathrm{Q}_\ell}(\beta_\ell),$$

and

$$\sigma(\alpha) = \prod_\ell \sigma_\ell(\alpha_\ell).$$

We may write $\beta = b \cdot \beta'$ for $b \in K^*$ where β'_v is "very close to 1" in K_v for all the problematic places v, there being finitely many of these; namely: all archimedean places and all places ramified over Q. Since $N_{K/\mathrm{Q}}(b) \in \mathrm{Q}^* \subset J_{\mathrm{Q}}$ is annihilated by σ, we now need only consider ideles β with β_v "very close to 1" in K_v for all the problematic places v.

Since the homomorphisms σ_ℓ are continuous we may assume that we have so modified β so that for all problematic ℓ's (i.e., the archimedean place and the ramified primes for K/Q) the contribution $\sigma_\ell(\alpha_\ell)$ acts trivially on K; it remains to see that if p is a finite unramified prime (for K/Q) $\sigma_p(\alpha_p)$ also acts trivially on K. Since $\alpha_p = \prod_{v \mid p} N_{K_v/\mathrm{Q}_p}\beta_v$ and since v is unramified, we have that $v_p(\alpha_p)$ is a multiple of $[K_v : \mathrm{Q}_p]$. Using the formula in subsection 5.3 we see that the image of $\sigma_p(\alpha_p)$ in $\mathrm{Gal}(K/\mathrm{Q})$ is a power of $Frob_p^{[K_v:\mathrm{Q}_p]}$; i.e., is trivial.

Corollary 3 *(***Global***) The surjective homomorphism σ in the above lemma induces an isomorphism of groups*

$$J_{\mathrm{Q}}/\mathrm{Q}^* \cdot N_{K/\mathrm{Q}} J_K \stackrel{\simeq}{\longrightarrow} \mathrm{Gal}(K/\mathrm{Q}).$$

Proof. Global Second inequality.

Corollary 4 *(***Local***) The homomorphism $\sigma_p : \mathrm{Q}_p^* \rightarrow \mathrm{Gal}(\mathrm{Q}_p(\mathrm{cycl})/\mathrm{Q}_p)$ induces an isomorphism of groups*

$$\mathrm{Q}_p^*/N_{K_v/\mathrm{Q}_p} K_v^* \stackrel{\simeq}{\longrightarrow} \mathrm{Gal}(K_v/\mathrm{Q}_p).$$

Proof. Local Second inequality (after a straightforward argument).

6.2 Local Reciprocity Homomorphisms over Q_p

Definition 7 *Let $p < \infty$. A* **Local Reciprocity Homomorphism (over Q_p)** *is a continuous homomorphism*

$$\mathrm{Q}_p^* \quad \xrightarrow{s} \quad \mathrm{Gal}(\mathrm{Q}_p(\mathrm{cycl})/\mathrm{Q}_p)$$

wtih dense image, and satisfying these two properties:

1. *For any intermediate extension $\mathcal{K}/\mathrm{Q}_p$ of finite degree in $\mathrm{Gal}(\mathrm{Q}_p(\mathrm{cycl})$ the kernel of the composite homomorphism*

$$\mathrm{Q}_p^* \quad \xrightarrow{s} \quad \mathrm{Gal}(\mathrm{Q}_p(\mathrm{cycl})/\mathrm{Q}_p) \quad \longrightarrow \quad \mathrm{Gal}(\mathcal{K}/\mathrm{Q}_p)$$

   is equal to the image of the norm mapping

$$N_{\mathcal{K}/\mathrm{Q}_p} : \mathcal{K}^* \longrightarrow \mathrm{Q}_p^*.$$

2. *The composition of the homomorphisms*

$$\mathrm{Q}_p^* \quad \xrightarrow{s} \quad \mathrm{Gal}(\mathrm{Q}_p(\mathrm{cycl})/\mathrm{Q}_p) \longrightarrow \quad \mathrm{Gal}(\mathrm{Q}_p(\text{cycl prime to } p)/\mathrm{Q}_p) \quad \simeq \quad \hat{\mathrm{Z}}$$

   is given by the valuation map,

$$\mathrm{Q}_p^* \longrightarrow \mathrm{Z} \subset \hat{\mathrm{Z}},$$

   i.e., $x \mapsto v_p(x)$ where $v_p(p) = 1$. (Here, of course, Z and $\hat{\mathrm{Z}}$ mean the additive topological group of the rings Z and $\hat{\mathrm{Z}}$.)

Lemma 5 *(***Uniqueness***) There is at most one Local Reciprocity Homomorphism over Q_p.*

Let s, s' be two such. It suffices to show that for *all* uniformizers $\pi \in \mathrm{Q}_p^*$, $s(\pi) = s'(\pi) \in \mathrm{Gal}(\mathrm{Q}_p(\mathrm{cycl})/\mathrm{Q}_p$. Let π be any such uniformizer. Let $\mathcal{L}, \mathcal{L}' \subset \mathrm{Q}_p(\mathrm{cycl})$ be the fixed subfields of the automorphisms $s(\pi)$ and $s'(\pi)$ respectively. Note that among the the subfields $\mathcal{K}$ of finite degree over Q_p, those that are contained in $\mathcal{L}$ are precisely the ones such that $N_{\mathcal{K}/\mathrm{Q}_p}\mathcal{K}^*$ contain π (by property (1) for s). But also those that are contained in $\mathcal{L}'$ are precisely the ones such that $N_{\mathcal{K}/\mathrm{Q}_p}\mathcal{K}^*$ contain π (by property (1) for s'). So, $\mathcal{L} = \mathcal{L}'$ which means that the (topological) cyclic groups that $s(\pi)$ and $s'(\pi)$ generate in $\mathrm{Gal}(\mathrm{Q}_p(\mathrm{cycl})/\mathrm{Q}_p)$ are equal. That is, $s'(\pi) = s(\pi)^u$ where $u \in \hat{\mathrm{Z}}^*$. Since s, s' both satisfy condition (2) we have that $u = 1$.

Corollary 5 *The homomorphism*

$$\mathrm{Q}_p^* \quad \xrightarrow{\sigma_p} \quad \mathrm{Gal}(\mathrm{Q}_p(\mathrm{cycl})/\mathrm{Q}_p)$$

in $\mathrm{Gal}(K_v/\mathrm{Q}_p)$ is equal to the Artin symbol. That is,

$$\sigma_p(x) \;=\; (x, \mathrm{Q}_p(\mathrm{cycl})/\mathrm{Q}_p).$$

Proof. Both σ_p and the Artin symbol satisfy the axioms of required for them to be Local Reciprocity Homomorphisms over Q_p.

7 Proof of Brauer reciprocity

Recall the definitions of Artin reciprocity (Definition 3) and of Brauer reciprocity(Definition 2).

Proposition 2 *Artin Reciprocity for cyclotomic fields over* Q *holds.*

Proof. Since, for any $a \in \mathrm{Q}^*$, $\prod_{\ell \leq \infty} \sigma_\ell(a_\ell) = 1$, Corollary 5 gives us that for all cyclotomic fields K/Q of finite degree, we have

$$\prod_{\ell \leq \infty} (a_\ell, K_v/\mathrm{Q}_\ell) = 1.$$

Corollary 6 *Brauer Reciprocity for cyclic cyclotomic fields over* Q *holds.*

Proof. This follows from the previous Proposition together with Lemma 2.

Corollary 7 *Any global L/K satisfies Brauer Reciprocity.*

Proof. Combine lemma 2 with Corollary 6.

8 Cohomology of idele class groups

We have, for any global extension (in our category) L/K, the exact sequence of $\mathrm{Gal}(L/K)$-modules

$$0 \rightarrow L^* \rightarrow J_L \rightarrow C_L \rightarrow 0$$

and (since the H^1's vanish) we have the induced exact sequence on H^2,

$$0 \rightarrow H^2(\mathrm{Gal}(L/K), L^*) \rightarrow H(L/K) \rightarrow H^2(\mathrm{Gal}(L/K), C_L) \rightarrow H^3(\mathrm{Gal}(L/K), L^*).$$

Proposition 3 *If* $\mathrm{Gal}(L/K)$ *is cyclic of order n, the above exact sequence becomes*

$$0 \rightarrow H^2(\mathrm{Gal}(L/K), L^*) \rightarrow H(L/K) \rightarrow H^2(\mathrm{Gal}(L/K), C_L) \rightarrow 0$$

and $H^2(\mathrm{Gal}(L/K), C_L)$ *is cyclic of order n. There is a (unique) isomorphism*

$$H^2(\mathrm{Gal}(L/K), C_L) \stackrel{inv_{L/K}}{\longrightarrow} \frac{1}{n}\mathrm{Z}/\mathrm{Z}$$

with the property that for all $c \in H(L/K)$ mapping to the cohomology class $\tilde{c} \in H^2(\mathrm{Gal}(L/K), C_L)$,

$$inv_{L/K}(\tilde{c}) := \quad I_{L/K}(c) \quad \in \quad \frac{1}{n}\mathrm{Z}/\mathrm{Z} \subset \mathrm{Q}/\mathrm{Z}.$$

Proof. Since $\mathrm{Gal}(L/K)$ is cyclic we have that $H^3(\mathrm{Gal}(L/K), L^*) \simeq H^1(\mathrm{Gal}(L/K), L^*)$ and this latter group vanishes by Hilbert's Theorem 90. By Corollary 3 we see that $\hat{H}^0(\mathrm{Gal}(L/K), C_L)$ is cyclic of order $[L : K]$ and therefore that $H^2(\mathrm{Gal}(L/K), C_L)$ is also cyclic of that order. By Brauer reciprocity (Corollary 7 above) we may take the above equation as the *definition* of $I_{L/K}(\tilde{c})$ since if $c, c' \in H(L/K)$ are two liftings of $\tilde{c}$ the difference $c' - c$ is a principal Brauer class and therefore $I_{L/K}(c' - c) = 0$ so $inv_{L/K}(\tilde{c}) \in \mathrm{Q}/\mathrm{Z}$ is well-defined. Clearly, $inv_{L/K}(\tilde{c})$ is of order dividing n, so lies in $\frac{1}{n}\mathrm{Z}/\mathrm{Z} \subset \mathrm{Q}/\mathrm{Z}$. Finally, we must show:

Lemma 6 *(Recall that L/K is assumed to be cyclic.) The homomorphism*

$$inv_{L/K} : H^2(\mathrm{Gal}(L/K), C_L) \quad \longrightarrow \quad \frac{1}{n}\mathrm{Z}/\mathrm{Z}$$

is an isomorphism.

Hint of Sketch of Proof: First reduce to n a power of a prime number ℓ. Then prove that there are (infinitely many) places v of K that remain prime in L. So: $[L_w : K_v] = [L : K]$. Choose one such place v, and construct the Brauer class $c \in H(L/K)$ having v-component $c_v \in H^2(Gal(L/K), L_w^*)$ with

$$inv_v(c_v) = \frac{1}{[L_w : K_v]} = \frac{1}{n}$$

and having trivial v'-component, i.e., $c_{v'} = 1$, for all $v' \neq v$. It follows that if $\tilde{c} \in H^2(\mathrm{Gal}(L/K), C_L)$ is the image of c, we have

$$inv_{L/K}(\tilde{c}) = \frac{1}{n} \quad \in \quad \frac{1}{n}\mathrm{Z}/\mathrm{Z},$$

and this proves our lemma.

Note that in general, $H(L/K) \to H^2(\mathrm{Gal}(L/K), C_L)$ is not necessarily surjective (!) so if we wish to extend our definition of $inv_{L/K}$ to extensions more general than cyclic ones, we need to do some "transporting of Brauer classes" as we hint at below:

Proposition 4 *Let L/K be any object of our category of degree n (i.e., finite Galois, with L of finite degree over Q). Then there is an isomorphism (which we continue to call $inv_{L/K}$)*

$$inv_{L/K} : H^2(\mathrm{Gal}(L/K), C_L) \quad \longrightarrow \quad \frac{1}{n}\mathrm{Z}/\mathrm{Z}$$

uniquely determined by the condition that there exists a cyclic Galois extension L'/K with that property for any element $h \in H^2(\mathrm{Gal}(L/K), C_L)$, there is a Brauer class $c' \in H(L'/K)$ such that $Inf_{L/K}^{LL'/L}(h) = Inf_{L'/K}^{LL'/L}\tilde{c}'$ with

$$inv_{L/K}(h) = inv_{L'/K}(\tilde{c}').$$

Note: We omit the proof of this here but there is hint at the method of proof in its formulation, and we note that Tate's file *"The subgroup $H_K \subset H^*(\ /K; C_K)$"* (next after this in the web-page of our course) sketches a proof of this.

Corollary 8 *Let L/K be any object of our category. We have a canonical exact sequence*

$$0 \to H^2(Gal(L/K), L^*) \to \bigoplus_{v \text{ place of } K} H^2(Gal(L_w/K_v), L_w^*) \overset{\text{sum of local invariants}}{\longrightarrow} \text{Q/Z}.$$

Thanks to all our previous work, this finishes what we needed to have to deduce:

Corollary 9 *The data $\{G := \text{Gal}(\bar{\text{Q}}/\text{Q}), \{G_K\}, A = C_K = J_K/K^*\}$ is a class formation and its invariant functional is*

$$inv_{L/K} : H^2(\text{Gal}(L/K), C_L) \quad \longrightarrow \quad \text{Q/Z}$$

as constructed in Proposition 4.

Corollary 10 *The homomorphism $J_k \to \mathcal{G}_K^{\text{ab}}$ given as teh product of local Artin symbols,*

$$\alpha \quad \longrightarrow \quad \prod_{v \text{ place of } K} (\alpha_v, K_v^{\text{ab}}/K_v) \quad \in \quad \mathcal{G}_K^{\text{ab}}$$

induces and isomorphism

$$C_K/\mathcal{N}_K \quad \overset{\simeq}{\longrightarrow} \quad \mathcal{G}_K^{\text{ab}}$$

where $\mathcal{N}_K := \cap_{L/K} N_{L/K} C_L \quad \subset \quad C_K$, the intersection of the norm groups being taken either over all finite degree L/K or over all abelian finite degree L/K (they have the same intersection).

What we have not done—to finish this chapter—is *the Global Existence Theorem*, i.e., we have not proved that $\mathcal{N}_K = D_K$, the connected component of C_K containing the identity element.