CATEGORY THEORY REFERENCE

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1. Basics

A category $\mathcal{C}$ is a collection of objects, a set of "homomorphisms" $\mathrm{Hom}_{\mathcal{C}}(A,B)$ for objects $A,B \in \mathcal{C}$, and a composition map $\circ : \mathrm{Hom}_{\mathcal{C}}(B,C) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{C}}(A,C)$. We also require the existence of a $1_A \in \mathrm{Hom}_{\mathcal{C}}(A,A)$ with $f \circ 1_A = f$ and $1_A \circ g = g$ for $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B,A)$. Furtherfore, composition should be associative. Examples of categories include groups with group homomorphisms as the maps, sets with arbitrary functions as map, and topological spaces with continuous functions as maps.

A covariant functor F from $\mathcal{C}$ to $\mathcal{D}$ sends objects of $\mathcal{C}$ to objects of $\mathcal{D}$, and sends $\mathrm{Hom}_{\mathcal{C}}(A,B)$ to $\mathrm{Hom}_{\mathcal{C}}(F(A),F(B))$. We require that $F(1_A) = 1_{F(A)}$ and $F(g \circ f) = F(g) \circ F(f)$ for $f \in \mathrm{Hom}_{\mathcal{C}}(A,A')$ and $g \in \mathrm{Hom}_{\mathcal{C}}(A',A'')$. A contravariant functor is similar except it reverses the direction of morphisms. An example is the field of fractions functor from the category of integral domains with injective maps to the category of fields which sends D to the field of fractions of D, and a map $\psi : D \to D'$ to the obvious map of the fields of fractions. $\mathrm{Hom}(A,-)$ is a covariant functor from any category to the category of sets. Given a map $B \xrightarrow{f} C$, $\mathrm{Hom}(A,f)$ sends $\phi \in \mathrm{Hom}(A,B)$ to $f \circ \phi \in \mathrm{Hom}(A,C)$.

Two functors can be related by natural transformations. A natural transformation η between functors F and G from $\mathcal{C}$ to $\mathcal{D}$ is a map η_A that makes the following diagram commute for any $A \in \mathcal{C}$ and $\psi \in \mathrm{Hom}_{\mathcal{C}}(A,B)$.

$$\begin{array}{ccc} F(A) & \xrightarrow{\eta_A} & G(A) \\ {\scriptstyle F\psi}\downarrow & & \downarrow{\scriptstyle G\psi} \\ G(A) & \xrightarrow{\eta_B} & G(B) \end{array}$$

If the maps η_A are isomorphisms (have inverses) for all A, the two functors are naturally isomorphic. For example, the functor sending a $G-$module to the set of points fixed by G is naturally isomorphic to the functor $\mathrm{Hom}_G(\mathbb{Z},-)$ with $\mathbb{Z}$ having the trivial $G-$action.

2. Abelian Categories

We now want additional structure in our categories so as to be able to talk about exact sequences. An additive category is one where $\mathrm{Hom}_{\mathcal{C}}(A,B)$ is an Abelian group, where the composition map is bi-additive (ie $h \circ (f+g) = h \circ f + h \circ g$, similarly in the other argument), and where every finite collection of objects has a direct sum (as characterized by the standard universal property). In an additive category it makes sense to talk about exact sequences.

$$0 \to A \to B \xrightarrow{\alpha} C$$

is exact if the sequence

$$0 \to \mathrm{Hom}(T, A) \to \mathrm{Hom}(T, B) \to \mathrm{Hom}(T, C)$$

is exact as a sequence of Abelian groups for any T. The object A then becomes the kernel of α. Likewise,

$$A \xrightarrow{\beta} B \to C \to 0$$

is exact if

$$0 \to \mathrm{Hom}(C, T) \to \mathrm{Hom}(B, T) \to \mathrm{Hom}(A, T)$$

is exact for any T, and C is the cokernel of β. In the category of $R-$modules, these behave exactly as expected. (There are also alternate characterization of objects as kernels and cokernels that do not need the additive structure.) We call a functor exact if it sends exact sequences to exact sequences.

An epimorphism is essentially a surjective function. $f : A \to B$ is an epimorphism if $g_1 \circ f = g_2 \circ f$ ($g_i \in \mathrm{Hom}(B, C)$) implies that $g_1 = g_2$. A monomorphism is like a injective function. $g : B \to C$ is a monomorphism if $g \circ f_1 = g \circ f_2$ implies $f_1 = f_2$. These differ from surjections and injections in some cases but for algebraic purposes agree (for example, consider embedding a metric space into its completion in the category of metric spaces with continuous maps. This map is an epimorphism but not surjective).

An additive category is an Abelian catgory if every morphism has a kernel and cokernel, every monomorphism is the kernel of some morphism, and every epimorphism is the cokernel of some morphism.

The main example of Abelian categories we've encountered so far is the category of $R-$modules. In fact, Mitchell's embedding theorem states that every small Abelian category embeds via a full, faithful, exact functor into the category of $R-$modules, so thinking about an Abelian category as $R-$mod really suffices for most purposes.

3. Derived Functors and Injective Resolutions

Let $\mathcal{C}$ be an Abelian category. An object I is injective if $\mathrm{Hom}(-, I)$ is an exact functor. An equivalent condition is that for every injection $A \hookrightarrow B$ with a map $A \to I$ the map extends to a one $B \to I$. For example, free modules are injective in the category of $R-$modules. An Abelian category has "enough injectives" if every object admits an map into an injective object with the mapping injective. In particular this lets us construct a long exact sequence of injective objects off of an arbitrary M:

$$0 \to M \to I^0 \to I^1 \to \dots$$

To do this, pick an injective map $d^0 : M \to I^0$ for some injective object I^0. Then let $B = \text{coker}(d_0)$, and pick an injective map $B \to I_1$ for some injective I_1. Then the composition $I_0 \to B \to I_1$ will keep the sequence exact. Continuing this way produces an injective resolution.

Now suppose we are given a functor F that is left exact. We wish to extend

$$0 \to F(A) \to F(B) \to F(C)$$

to a longer exact sequence. To do this, pick an injective resolution of each object, and apply F to this. This is no longer necessarily an exact sequence, so look at the cohomology groups $H^r(F(I^\circ))$. It turns out that no matter what injective resolution we pick for A these cohomology groups will be naturally isomorphic. Therefore define $(R^rF)(A) = H^r(F(I^\circ))$. A morphism $A \to B$ then gives a map $R^rF(A)$ to $R^rF(B)$ by looking at the induced map of chains, so R^rF is a functor. It is called the right derived functors of F. The most important result about them is that if

$$0 \to A \to B \to C \to 0$$

is exact then

$$0 \to F(A) \to F(B) \to F(C) \to (R^1F)(A) \to (R^1F)(B) \to (R^1F)(C) \to (R^2F)(A) \dots$$

is an exact sequence.

An alternate axiomatic characterization of the right derived functors is that $(R^0F) = F$, $(R^nF)(I) = 0$ for injective objects I and $r > 0$, and that they fit in the above long exact sequence.

The group cohomology functors we've been dealing with are the derived functors for the Hom functor. $\mathcal{H}^n(G, -)$ is the n^{th} derived functor for $\text{Hom}_{\mathbb{Z}G}(\mathbb{Z}, -)$.

An alternate approach involves constructing projective resolutions and applying contravariant functors like $\text{Hom}(-, A)$ to get the cohomology groups.