

Introduction to Dirichlet series and the Dedekind zeta function

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These notes are meant to give an introduction to the basic analysis behind Dirichlet series and the Dedekind zeta function. They include a description of the conditions which make Dirichlet series converge uniformly to continuous functions, using Abel's summation formula, computing a residue at $x = 1$ of the Riemann zeta function, defining an Euler product of Dirichlet series, and concluding with a definition of the Dedekind zeta function, proving its convergence and Euler product formula, and stating a theorem describing its residue at $x = 1$.

Dirichlet series and Abel summation

A general Dirichlet series is defined for a sequence $\{a_i\}$ of complex numbers to be $\sum_{i=1}^{\infty} a_i i^{-x}$. We can prove the following proposition regarding its convergence:

Proposition. *If $\sum_{i=1}^{\infty} a_i i^{-x}$ converges absolutely at x_0 , then it converges uniformly on $[x_0, \infty)$.*

Proof. Take $x > x_0$, then

$$\left| \frac{a_i i^{-x}}{a_i i^{-x_0}} \right| = n^{x_0 - x}$$

converges to zero as n goes to infinity. Since the absolute value of the coefficients of $\sum_{i=1}^{\infty} a_i i^{-x}$ and $\sum_{i=1}^{\infty} a_i i^{-x_0}$ converges, the comparison test tells us that $\sum_{i=1}^{\infty} a_i i^{-x}$ converges absolutely. To show uniform convergence, notice

$$0 \leq |a_i i^{-x}| \leq |a_i i^{-x_0}|,$$

so that each term $|a_i i^{-x}|$ is bounded by a constant M_i such that $\sum M_i$ converges. This demonstrates uniform convergence on the interval $[x_0, \infty)$, since the convergence of the partial sums $\sum_{i=1}^n |a_i i^{-x}|$ is thus controlled independently of x . This is also called the Weierstrass M-test.

Because a uniform limit of continuous functions is continuous, the Dirichlet series on this interval is thus continuous (it is a uniform limit of its partial sums, each of which is continuous). \square

Now we introduce Abel summation, which will be useful for Dirichlet series.

Abel summation. *Let $\{a_i\}$, $\{b_i\}$ be sequences, let $A_n = \sum_{i=m+1}^n a_i$. Then*

$$\sum_{m+1}^n a_i b_i = A_n b_{n+1} - A_m b_{m+1} - \sum_{m+1}^n A_i (b_{i+1} - b_i)$$

Proof.

$$\begin{aligned}
\sum_{i=m+1}^n a_i b_i &= \sum_{i=m+1}^n (A_i - A_{i-1}) b_i \\
&= \sum_{i=m+1}^n A_i b_i - \sum_{i=m+1}^n A_{i-1} b_i \\
&= \sum_{i=m+1}^n A_i b_i - \left(\sum_{i=m+1}^n A_i b_{i+1} + A_m b_{m+1} - A_n b_{n+1} \right) \\
&= A_n b_{n+1} - A_m b_{m+1} - \sum_{i=m+1}^n A_i (b_{i+1} - b_i)
\end{aligned}$$

□

Before applying Abel summation to Dirichlet series, we prove the following lemma.

Lemma. *Given $\{a_i\}$, $\{b_i\}$ sequences, assume $|A_n|$ is bounded and $\{b_i\}$ is monotonically decreasing, converging to zero. Then $\sum_{i=1}^{\infty} a_i b_i$ converges.*

Proof. Use Abel summation and the absolute value inequalities:

$$\left| \sum_{i=m+1}^n a_i b_i \right| \leq |A_n| b_{n+1} + |A_m| b_{m+1} + \sum_{i=m+1}^n |A_i| (b_i - b_{i+1}),$$

then if M is the bound for $|A_n|$, and for m, n sufficiently large, this expression is less than

$$\begin{aligned}
M(b_{n+1}) + M(b_{m+1}) + M \sum_{i=m+1}^n (b_i - b_{i+1}) &= M(b_{n+1} + b_{m+1} + b_{m+1} - b_{n+1}) \\
&\leq M(\epsilon/2M + \epsilon/2M) = \epsilon
\end{aligned}$$

Hence the partial sums $|\sum_{i=m+1}^n a_i b_i|$ form a Cauchy sequence and hence converge as desired. □

We now apply the Abel business to Dirichlet series.

Proposition. *Assume the partial sums of the sequence $\{a_i\}$, $|\sum_{i=1}^n a_i|$ are bounded for each n . Then $\sum_{i=1}^{\infty} a_i i^{-x}$ converges uniformly on $[y, \infty)$ for all $y > 0$, and hence defines a continuous function on $(0, \infty)$.*

Proof. Letting $b_i = i^{-y}$ for $y > 0$ we note that $\sum_{i=1}^{\infty} a_i i^{-y}$ converges on $y > 0$ since the conditions of the Lemma are fulfilled.

Now take $x \geq 0$ and let $a'_i = a_i i^{-y}$, $b'_i = i^{-x}$, $A'_i = \sum_{i=m+1}^n a'_i$. Then we have

$$\begin{aligned}
\left| \sum_{i=m+1}^n a_i i^{-(x+y)} \right| &\leq |A'_n| (n+1)^{-x} + |A'_m| (m+1)^{-x} + \sum_{i=m+1}^n |A'_i| (i^{-x} - (i+1)^{-x}) \\
&\leq (\epsilon/2)(n+1)^{-x} + (\epsilon/2)(m+1)^{-x} + \sum_{i=m+1}^n (\epsilon/2)(i^{-x} - (i+1)^{-x}) \\
&= (\epsilon/2)((n+1)^{-x} + (m+1)^{-x} + (m+1)^{-x} - (n+1)^{-x}) \\
&\leq 2\epsilon/2 = \epsilon,
\end{aligned}$$

where the second inequality assumes m, n sufficiently large that $|A'_n| < \epsilon/2$, which is valid since $\sum_{i=1}^{\infty} a'_i$ converges. The last inequality is due to the fact that $(m+1)^{-x} \leq 1$ when $x \geq 0$. This shows the convergence on $[y, \infty)$ is uniform since it does not rely on y . The continuity of the Dirichlet series again follows from the uniformity of convergence. □

The Riemann zeta function at $x = 1$

The previous proposition then shows that $\sum_{i=1}^{\infty} (-1)^i i^{-x}$ converges to a continuous function on $(0, \infty)$. However the Riemann zeta function $\sum_{i=1}^{\infty} i^{-x}$ does not fulfill the hypotheses of the proposition, but is convergent and continuous on $(1, \infty)$ by our very first proposition. We will now compute the residue of the Riemann zeta function at $x = 1$.

Proposition. *Let $\phi(x) = \zeta(x) - \frac{1}{x-1}$, then ϕ is continuous on $(1, \infty)$ and $0 \leq \phi(x) \leq 1$ for x in this range.*

Proof. Continuity follows from the continuity of the Riemann zeta function on $(1, \infty)$.

To show the second part of the theorem, take $x > 1$, then for $n \geq 1$, we have

$$(n+1)^{-x} \leq \int_n^{n+1} y^{-x} dy \leq n^{-x},$$

since for $n \leq y \leq n+1$ we have $(n+1)^{-x} \leq y^{-x} \leq n^{-x}$, and the interval of integration has length 1. Then summing over all n we have

$$\begin{aligned} \zeta(x) - 1 &\leq \int_1^{\infty} y^{-x} dy \leq \zeta(x) \\ \zeta(x) - 1 &\leq \frac{1}{x-1} \leq \zeta(x), \end{aligned}$$

which shows $0 \leq \phi(x) \leq 1$. □

Corollary.

$$\lim_{x \rightarrow 1^+} (x-1)\zeta(x) = 1$$

Proof. Since ϕ takes values in $[0, 1]$, $\lim_{x \rightarrow 1^+} (x-1)\phi(x) = 0$, so $\lim_{x \rightarrow 1^+} (x-1)(\zeta(x) - \frac{1}{x-1}) = 0$ and thus $\lim_{x \rightarrow 1^+} (x-1)\zeta(x) = 1$. □

Euler products for Dirichlet series

First we recall some facts about infinite products. $\prod_{i=1}^{\infty} (1 + b_i)$ for $b_i \neq -1$ is convergent if the partial products $\prod_{i=1}^n (1 + b_i)$ are convergent. Similarly to sums, an infinite product converges absolutely when $\prod_{i=1}^{\infty} (1 + |b_i|)$ is convergent. Re-ordering of an absolutely convergent product does not change its value and an absolutely convergent product is convergent.

Lemma. $\prod_{i=1}^{\infty} (1 + |b_i|)$ converges iff $\sum_{i=1}^{\infty} |b_i|$ converges.

Proof. Expanding partial products we have the inequality $\prod_{i=1}^n (1 + |b_i|) \geq \sum_{i=1}^n |b_i|$, proving one direction of the lemma.

To prove the other direction, note that for $x \geq 0$, $e^x \geq 1 + x$ by its power series expansion. Then $e^{\sum_{i=1}^n |b_i|} \geq \prod_{i=1}^n (1 + |b_i|)$ and if the partial sums converge, then so do the partial products. □

A multiplicative sequence is a sequence $\{a_i\}$ such that $a_i a_j = a_{ij}$ when i, j are relatively prime.

Theorem. *The following two conditions are equivalent:*

1. $\sum_{i=1}^{\infty} a_{p^i} p^{-ix}$ converges absolutely and $\prod_{i=1}^{\infty} (1 + \sum_{i=1}^{\infty} |a_{p^i}| p^{-ix})$ converges
2. $1 + \sum_{i=2}^{\infty} a_i i^{-x}$ converges absolutely.

Proof. (1 \Leftrightarrow 2) Since $\sum_{i=1}^{\infty} a_{p^i} p^{-ix}$ is a subseries of $1 + \sum_{i=2}^{\infty} a_i i^{-x}$, it converges absolutely. Now let S be a finite set of primes, then we have the equality

$$\prod_{p \in S} (1 + \sum_{i=1}^{\infty} |a_{p^i}| p^{-ix}) = 1 + \sum_{i \geq 2, S \ni p|i} a_i i^{-x}$$

(where the sum on the right hand side is taken over all i such that $p|i$ for some $p \in S$), since expanding the product will give us products of infinite series which, for two different primes $p \neq p'$ look like

$$\left(\sum_{i=1}^{\infty} |a_{p^i}| p^{-ix}\right) \left(\sum_{i=1}^{\infty} |a_{p'^i}| p'^{-ix}\right) = \sum_{i=1}^{\infty} \sum_{d \leq i} |a_{p^d} a_{p'^{i-d}}| p^{-dx} p'^{-(i-d)x},$$

and since the sequence is multiplicative, this is

$$\sum_{i=1}^{\infty} \sum_{d \leq i} |a_{p^d} a_{p'^{i-d}}| (p^d p'^{(i-d)})^{-x},$$

and expanding the finite product will give all combinations of powers of primes in S , yielding the sum on the right. But this sum $1 + \sum_{i \geq 2, S \ni p|i} a_i i^{-x}$ is bounded above by assumption, so that all the partial products are bounded, and thus the product $\prod_p (1 + \sum_{i=1}^{\infty} |a_{p^i}| p^{-ix})$ converges.

(1 \Rightarrow 2) We use a similar argument to prove the opposite direction. Let S_m be the set of primes dividing an integer m . Then we have

$$\begin{aligned} \prod_p \left(1 + \sum_{i=1}^m |a_{p^i}| p^{-ix}\right) &\leq 1 + \sum_{i \geq 2, S_m \ni p|i} a_i i^{-x} \\ &= \prod_{p \in S_m} \left(1 + \sum_{i=1}^{\infty} |a_{p^i}| p^{-ix}\right) \\ &\leq \prod_p \left(1 + \sum_{i=1}^{\infty} |a_{p^i}| p^{-ix}\right), \end{aligned}$$

which converges by assumption. □

Using the equality from the previous proof,

$$1 + \sum_{i \geq 2, S_m \ni p|i} a_i i^{-x} = \prod_{p \in S_m} \left(1 + \sum_{i=1}^{\infty} |a_{p^i}| p^{-ix}\right).$$

Then assuming the two equivalent convergence conditions of the previous theorem hold, these two partial sums converge to the same thing as $m \rightarrow \infty$.

The Dedekind zeta function and its Euler product

If K is a number field of finite degree over \mathbb{Q} , we defined the Dedekind zeta function of K to be

$$\zeta_K(x) = \sum_{I \subset \mathcal{O}_K} N I^{-x},$$

where the sum ranges over all nonzero integral ideals I .

We consider when this series converges and prove its Euler product representation.

Theorem. $\zeta_K(x)$ converges on $(1, \infty)$ and on this interval,

$$\zeta_K(x) = \prod_P (1 - N P^{-x})^{-1},$$

where P denotes a prime ideal of \mathcal{O}_K .

Proof. First note that $1 - (NP^{-x})^2 < 1$ and that $1 + NP^{-x} - 2NP^{-2x} \geq 1$ (the second fact is true since $NP^{-x} \leq 1/2$, so $2NP^{-2x} \leq NP^{-x}$). This gives us the inequalities $1_N P^{-x} < \frac{1}{1 - NP^{-x}} \leq 1 + 2NP^{-x}$.

Now let $(1 - NP^{-x})^{-1} = 1 + b_p$, for $P|(p)$, so that $b_p = (1 - NP^{-x})^{-1} - 1 \leq 1 + 2NP^{-x} - 1 = 2NP^{-x} = 2p^{-fx}$ where f is the residual degree of P . Then

$$\sum_p b_p \leq 2[K : \mathbb{Q}] \sum_p p^{-x} \leq 2[K : \mathbb{Q}] \sum_n n^{-x},$$

since $[K : \mathbb{Q}]$ is the number of primes P above (p) (this is because $[K : \mathbb{Q}] = \sum_{i=1}^g e_i f_i$). Since $x > 1$ this series converges. And in fact, $b_n > 1 + NP^{-x} - 1 > 0$, so that $|b_n| = b_n$ so the series is absolutely convergent.

Now using a geometric series expansion, $\frac{1}{1 - NP^{-x}} = 1 + \sum_P NP^{-ix}$. Then it is left to show that, if S_m denotes the finite set of prime ideals P with $NP \leq m$,

$$\prod_{P \in S_m} (1 + \sum_P NP^{-ix}) = 1 + \sum_{I \subset \mathcal{O}_K, S_m \ni P|I} NI^{-x}.$$

But this is true by the exact same reasoning as in proof of the Euler product theorem for Dirichlet series in the previous subsection, where the coefficients a_i are 1 in this case, and we deal with multiplication of prime ideals rather than prime integers. Unique factorization of ideals and multiplicativity of the norm make the argument the same, giving this equality for all m . Since both sides converge as $m \rightarrow \infty$ by the same lemma in the previous subsection, the two limits of the two sides are equal. \square

The Riemann zeta function can be considered a special case of the Dedekind zeta function, in particular it is $\zeta_{\mathbb{Q}}(x)$ since the norm of a rational prime ideal (p) is p . In considering the residue of the general Dedekind zeta function at $x = 1$, we discover it contains arithmetic information about the number field K (which in the case of \mathbb{Q} is uninteresting). To conclude these notes, we state, but do not prove, the following theorem:

Theorem.

$$\lim_{x \rightarrow 1^+} (x - 1)\zeta_K(x) = \frac{2^{s+t} \pi^t R_K h_K}{W_K |d_K|^{1/2}}.$$

In this theorem, s and $2t$ are the number of real and imaginary imbeddings of K , respectively, h_K is the order of the ideal class group of K , R_K is the regulator of K , W_K is the number of roots of unity contained in K , and d_K is the field discriminant of K .