Differential Geometry 230ar

Final Exam

Due: Friday January 14th, 2005

Attempt at least five of the following problems. If you have questions, email me: weinkove@math.harvard.edu. Hand in to my mailbox next to the Math office on the 3rd floor.

Problems marked with (*) are very difficult (if not impossible) open problems. Don't try them until you have done five other problems.

1. Show that all curvature tensors vanish everywhere on any Riemannian manifold of dimension 1.

2. Let (M, g) be a compact Riemannian manifold of dimension n. Consider the parabolic flow of smooth functions f on M given by

$$\frac{\partial f}{\partial t} = \triangle f + f^2,$$

where f is initially a positive function. By calculating the evolution of the integral

$$\int_M f dV,$$

show that the flow cannot admit a smooth solution for all time. Here,

$$dV = \sqrt{\det g}\, dx^1 \wedge \cdots \wedge dx^n$$

is the standard volume form.

(You may find the following simple fact useful: if a function $y = y(t)$ for $t \in [0, T)$ satisfies the differential inequality $y'(t) \geq F(y)$ for some increasing function F and if $z(t)$ is a solution of the ODE $z'(t) = F(z)$ on $[0, T)$ with $z(0) = y(0)$ then $y(t) \geq z(t)$ for $t \in [0, T)$.)

3. Let (M, ω) be a compact Kähler manifold. As usual we write

$$\omega = \frac{\sqrt{-1}}{2} g_{i\bar{j}} dz^i \wedge d\bar{z}^j.$$

Suppose that $\omega \in \lambda c_1(M)$ for some $\lambda \in \mathbf{R}$, and that the metric g has constant scalar curvature. Show that g is in fact a Kähler-Einstein metric.

4. Consider the unnormalized Kähler-Ricci flow

$$\frac{\partial}{\partial t} g'_{i\bar{j}} = -R'_{i\bar{j}}$$

on a compact Kähler manifold M with $g'(0) = g$, for some fixed Kähler metric g on M. Show that as long as the flow exists, the scalar curvature R' of the metric g' satisfies

$$R' \geq -C,$$

for a constant C depending only on (M, g).

5. Let M be a compact Kähler manifold, with Kähler metric ω.

   (a) Show that
   $$n\mathrm{Ric}(\omega) \wedge \omega^{n-1} = R\omega^n,$$
   where we are using the usual notation.

   (b) Show that if ω and ω' are any two metrics in the same Kähler class, then
   $$\int_M R' dV' = \int_M R dV,$$
   where dV and dV' are the volume forms with respect to the metrics ω and ω'.

   (c) Show that, for a Riemann surface, part (b) holds even if the classes of ω and ω' differ by a constant multiple.
   (In fact, on a Riemann surface, the integral $\int_M R dV$ is a topological constant - it is given by the Gauss Bonnet formula - but don't use this.)

6. Suppose that M is a compact complex manifold with $c_1(M) > 0$. Fix a Kähler metric $\omega \in \pi c_1(M)$. Write $\omega = (\frac{\sqrt{-1}}{2}) g_{i\bar{j}} dz^i \wedge d\bar{z}^j$. Let f be a real function satisfying

$$R_{i\bar{j}} = g_{i\bar{j}} + \partial_i \partial_{\bar{j}} f.$$

Let g' be another Kähler metric in the same class as g and write

$$g'_{i\bar{j}} = g_{i\bar{j}} + \partial_i \partial_{\bar{j}} u.$$

Consider the flow of potentials $u = u(x, t)$ given by

$$\frac{\partial u}{\partial t} = \log \left(\frac{\det (g'_{i\bar{j}})}{\det (g_{i\bar{j}})} \right) + u - f,$$

with $u(0) = 0$. Assume that a solution exists on $M \times [0, T)$ for some fixed $T > 0$.

(a) Show that on $M \times [0, T)$, $g'_{i\bar{j}}$ satisfies

$$\frac{\partial}{\partial t} g'_{i\bar{j}} = -R'_{i\bar{j}} + g'_{i\bar{j}},$$

where $R'_{i\bar{j}}$ is the Ricci curvature tensor of the metric g' (i.e. this flow is the normalized Kähler-Ricci flow.)

(b) Show that there exist constants A and C depending only on the initial data (M, g) and f (and not on T) such that for any $(x, t) \in M \times [0, T)$,

$$(n + \triangle u)(x, t) \leq C e^{Au(x,t) - (A+1) \inf_{M \times [0,t]} u}.$$

7. Show that in the case $c_1(M) > 0$ as above, the normalized Kähler-Ricci flow exists for all time. (Note that the flow may still blow up as $t \to \infty$.) You can use the estimate in part (b) of the last problem, even if you didn't manage to prove it yourself. Also, there is no need to prove every detail - an outline of the proof is fine.

   (Hint: obtain estimates depending on time.)

8. (*) Again, in the case $c_1(M) > 0$, prove uniform estimates for the potential u and its derivatives assuming only that the initial metric has bisectional curvature satisfying

$$R_{i\bar{j}k\bar{l}} X^i \overline{X^j} Y^k \overline{Y^l} > 0$$

   for all vectors X and Y. It is known by the work of Bando and Mok that this condition is preserved along the flow. You may not use the solution of the Frankel conjecture (solved by Mori and Siu-Yau) that a manifold with this condition is biholomorphic to $\mathbf{CP}^n$. Also, you may not assume that M admits a Kähler-Einstein metric (with this assumption, X.X. Chen and Tian have proved convergence of the flow.)

9. Let (M, ω) be a compact Kähler manifold. The *Calabi flow* is a fourth order parabolic flow of potentials u given by

$$\frac{\partial u}{\partial t} = R' - \underline{R},$$

where $g'_{i\bar{j}} = g_{i\bar{j}} + \partial_i \partial_{\bar{j}} u$, R' is the scalar curvature of g' and $\underline{R}$ is the constant given by

$$\underline{R} = \frac{1}{\mathrm{Vol}(M)} \int_M R dV.$$

(a) Calculate $\dfrac{\partial R'}{\partial t}$.

(b) Calculate $\dfrac{\partial}{\partial t}(\omega'^n)$.

(c) Show that the square of the L^2 norm of the scalar curvature,

$$\int_M R'^2 dV',$$

is decreasing along the Calabi flow.

10. Hamilton's Ricci flow is the flow of Riemannian metrics on a compact manifold M given by

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}.$$

One of the key contributions of Perelman to the study of the Ricci flow is his introduction of the entropy functional

$$\mathcal{F}(g, f) = \int_M (R + |\nabla f|^2) e^{-f} dV,$$

where g is a Riemannian metric and f is a smooth function on the manifold. Note that the quantities R, $|\nabla f|^2$ and dV, the standard volume form, refer to the metric g. Show that if $g(t)$ satisfies the Ricci flow and if f satisfies the *backwards* heat-type equation

$$\frac{\partial f}{\partial t} = -\triangle f + |\nabla f|^2 - R,$$

(where $\triangle f$, $|\nabla f|^2$ and R are calculated with respect to the varying metric $g(t)$) then

$$\frac{\partial}{\partial t} \mathcal{F} = 2 \int_M |R_{ij} + \nabla_i \nabla_j f|^2 e^{-f} dV.$$

Hence the entropy functional $\mathcal{F}$ increases in time.

You may want to look at Perelman's paper on the preprint server arxiv.org/abs/math/0211159. But don't just copy his argument. You must show the detailed calculations.

(Aside: since f satisfies a backwards heat equation it is not parabolic and we cannot find a solution in general. But if you fix a point in time, then we can solve it backwards - since then it does become parabolic. The strength of the functional lies in the fact that you are completely free to choose f at any point in time, and can choose it so that it concentrates at a certain point. This allows you to obtain local information about the flow and show that the manifold does not 'collapse'.)

11. (*) Find a Perelman-type entropy functional for the Calabi flow (problem 9). Note that f will probably have to solve a backwards *fourth order* heat equation.