Differential Geometry 230ar

Homework 1

Due: Wednesday October 13th

Please attempt all of the problems. But you only need to hand in five of them, and you'll receive full credit if you answer all five correctly.

1. Let $\{(V_\alpha, \psi_\alpha)\}$ be a covering of M by C^∞-compatible coordinate charts. Show that the collection $\mathcal{U}$ of coordinate charts given by

$$\mathcal{U} = \{(U, \phi) \mid (U, \phi) \text{ is } C^\infty\text{-compatible with every } (V_\alpha, \psi_\alpha)\}$$

defines a differentiable structure on M.

2. Consider the unit sphere

$$S^2 = \{(x^1, x^2, x^3) \in \mathbf{R}^3 \mid (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\}.$$

Let $N = (0, 0, 1)$ and let $S = (0, 0, -1)$. Define

$$\begin{aligned} U_N &= \{x \in S^2 \mid x \neq N\} \\ U_S &= \{x \in S^2 \mid x \neq S\}. \end{aligned}$$

Let ϕ_N (ϕ_S) be the stereographic projection from N (S) to the x^1x^2-plane. Show that the charts (U_N, ϕ_N) and (U_S, ϕ_S) make S^2 into a differentiable manifold.

3. Let M be a smooth n dimensional manifold. Fix a point p in M which lies in a coordinate chart (U, ϕ), where $\phi = (x^1, \cdots x^n)$. Recall that the tangent space to M at p is given by

$$T_pM = \{V_p : C^\infty(p) \to \mathbf{R} \mid V_p \text{ is a derivation}\}.$$

(This means that V_p is a linear operator and satisfies the Leibniz rule.) Show that every V_p in T_pM can be written

$$V_p = \sum_{i=1}^{n} V^i (\phi^{-1})_* (\frac{\partial}{\partial x^i})|_p,$$

for some $V_i \in \mathbf{R}$.

Recall that $(\phi^{-1})_*$ is defined by

$$\left((\phi^{-1})_* (\frac{\partial}{\partial x^i}) \right) f = \frac{\partial}{\partial x^i} (f \circ \phi^{-1}).$$

4. The *rank* of a tensor $A = A^{ij}\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$ at a point x on a manifold is defined to be the rank of the matrix $(A^{ij}(x))$.

   (a) Why is the rank of A independent of the choice of coordinate system?

   (b) Let $V = V^i\frac{\partial}{\partial x^i}$ and $W = W^i\frac{\partial}{\partial x^i}$ be vectors. Find the rank of the tensors V^iW^j and $(V^iW^j + V^jW^i)$. Your answers may depend on the choice of vectors V and W.

5. Let $X = X^i\frac{\partial}{\partial x^i}$ and $Y = Y^i\frac{\partial}{\partial x^i}$ be vector fields on M. Recall that we define the *Lie bracket* $[X, Y]$ to be the vector field acting on functions $f \in C^\infty(M)$ by

$$[X, Y](f) = X(Y(f)) - Y(X(f)).$$

   Show that in local coordinates the Lie bracket is given by

$$[X, Y] = X^i\frac{\partial Y^j}{\partial x^i}\frac{\partial}{\partial x^j} - Y^i\frac{\partial X^j}{\partial x^i}\frac{\partial}{\partial x^j}.$$

6. The *Christoffel symbols* corresponding to a metric g_{ij} on a manifold M are given by

$$\Gamma^l_{ij} = \frac{1}{2}g^{kl}(\frac{\partial}{\partial x^i}g_{jk} + \frac{\partial}{\partial x^j}g_{ik} - \frac{\partial}{\partial x^k}g_{ij}).$$

   (a) Show that the Γ^l_{ij} are not components of a tensor.

   (b) Let Γ^l_{ij} and $\overline{\Gamma}^l_{ij}$ be the Christoffel symbols corresponding to metrics g_{ij} and $\overline{g}_{ij}$ respectively. Show that

$$L^l_{ij} = \Gamma^l_{ij} - \Gamma^l_{ij}$$

   defines a tensor on M.

7. The *covariant derivative* ∇S of a tensor S with components $S^{i_1\cdots i_r}_{j_1\cdots j_s}$ is defined by the formula

$$\begin{aligned}\nabla_k S^{i_1\cdots i_r}_{j_1\cdots j_s} &= \frac{\partial}{\partial x^k}S^{i_1\cdots i_r}_{j_1\cdots j_s} + \Gamma^{i_1}_{kl}S^{li_2\cdots i_r}_{j_1\cdots j_s} + \cdots + \Gamma^{i_r}_{kl}S^{i_1\cdots i_{r-1}l}_{j_1\cdots j_s} \\ &\quad -\Gamma^l_{kj_1}S^{i_1\cdots i_r}_{lj_2\cdots j_s} - \cdots - \Gamma^l_{kj_s}S^{i_1\cdots i_r}_{j_1\cdots j_{s-1}l}.\end{aligned}$$

   Using this formula show that

$$\nabla_k g_{ij} = 0.$$

8. Let $\gamma : (a, b) \to 0$ be a geodesic. That is, if $\gamma = \gamma(t)$ is given in local coordinates by $(x^1(t), \ldots, x^n(t))$ then the $x^i(t)$ locally satisfy

$$\frac{d^2 x^k}{dt^2} + \Gamma^k_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} = 0.$$

Show that

$$\frac{d}{dt} |\dot{\gamma}| = 0,$$

where $|\dot{\gamma}|$ is the speed of γ, given in local coordinates by the formula

$$|\dot{\gamma}| = \sqrt{g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}}.$$

9. Show that geodesics are critical points of the length functional

$$L(\gamma) = \int_a^b |\dot{\gamma}| dt,$$

in the following sense. Suppose for some $\delta > 0$, that

$$\gamma_s : (a - \delta, b + \delta) \to M$$

is a family of curves with $\gamma_s(a) = p$ and $\gamma_s(b) = q$ for p and q fixed points in the same coordinate patch U. Suppose that γ_0 is a geodesic. Then show that

$$\frac{d}{ds} L(\gamma_s)|_{s=0} = 0.$$